Lilian Contius, Cambridge Open Internship 2025

Combinatorics of KP Solitons

One of the beautiful things about mathematics is how intertwined seemingly distant areas of research can be, and this project is a great example for this. We use the methods of tropical geometry and combinatorics to study the solutions of the KP-equation, a PDE that models shallow water waves.

In 1970, Kadomtsev and Petviashili proposed the KP-equation to model the non-linear motion of shallow water waves in two spatial dimensions. Solutions to this equation are given by the function $u(x, y, t)$ which gives the amplitude of the wave at a certain point:

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + 3\frac{\partial^2 u}{\partial y^2} = 0, \qquad u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\ln \tau(x, y, t)$$

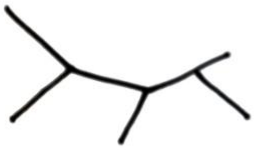
KP-equation and its solutions

The function $\tau(x, y, t)$ is determined by a set of n affine functionals, $\theta_i(x, y, t)$, defined by distinct real parameters κ_i. Summing over k-sized subsets of these functionals gives the exponential terms in $\tau(x, y, t)$:

$$\theta_i(x, y, t) = \left(\sum_{j=1}^{m} \kappa_i^j\, t_j\right) + c_i, i \in \{1, \dots, n\}, \qquad \tau(x, y, t) = \sum_{I \in \binom{[n]}{k}} C_I \exp\left(\sum_{i \in I} \theta_i(x, y, t)\right)$$

Affine functionals and $\tau(x, y, t)$ (C_I are constants)

To study solutions, we introduce a tropical function which approximates $\ln \tau(x, y, t)$. It only considers the maximal term in the outer sum of $\tau(x, y, t)$. Rescaling parameters yields the tropical function below. Points where the wave has non-zero amplitude correspond to points where this maximum is achieved at least twice. These points form what is known as a tropical variety. In two spatial dimensions, this is a branched structure approximating crests of the waves, shown on the right.



Sample tropical variety for parameters (x,y,t), approximates the crests of waves (m=3)

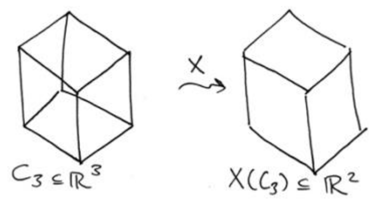
$$f(\bar{x}, \bar{y}, a) = \max_{I} \sum_{i \in I} \theta_i(\bar{x}, \bar{y}, \boldsymbol{a})$$

Tropical approximation of $\ln \tau(x, y, t)$. $(\bar{x}, \bar{y}, \boldsymbol{a})$ are rescaled parameters

The KP-equation is part of a hierarchy of PDEs known as the KP-hierarchy, where the "higher" PDEs include additional parameters. Setting $x = t_1, y = t_2$ and replacing t by $\boldsymbol{t} = (t_3, t_4, \dots, t_m)$ generalises the solutions $u(x, y, \boldsymbol{t})$ to arbitrarily many dimensions. It turns out that these solutions have a rich combinatorial structure. To explore this, we introduce the combinatorial object of tiled zonotopes. We can think of zonotopes as projections of regular n-cubes, where X is some $m \times n$ matrix projecting the cube into $\mathbb{R}^m$. We call them "tiled" if they have a subdivision into smaller zonotopes.

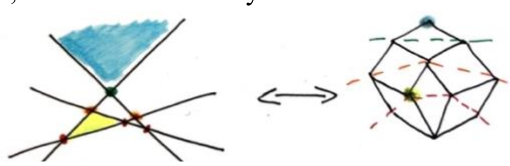
$$Z(X) = X(C_n) = X([-1,1]^n) \subseteq \mathbb{R}^m$$

Definition of a zonotope as projection of a regular n-cube



Tiled zonotope from projection of 3-cube, n=3,m=2

Plotting our n affine functionals by $t_0 = \theta_i(x, y, t)$ gives a hyperplane arrangement in $\mathbb{R}^{m+1}$ and the tropical varieties we are interested in correspond to subsets of this hyperplane arrangement. It can be shown that every hyperplane arrangement has a dual tiled zonotope, defined by an order-reversing bijection. This for example sends regions of the hyperplane arrangement to vertices of the zonotope, shown in blue and yellow on the diagram. Since these objects are combinatorially equivalent, we can represent the tropical varieties that approximate crests of the waves by parts of the tiled zonotope. We refer to these as cross-sections of the tiled zonotope. These results unify the approaches of Dimakis and Müller-Hoissen[1] and Kodama and Williams[2], who restricted their descriptions to specific values of k and m respectively.



Bijection between hyperplane arrangements and tiled zonotopes, m=2. For m=2, tropical varieties are points on the arrangement and cross-sections are dotted lines on the zonotope.

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[1] Aristophanes Dimakis and Folkert Müller-Hoissen. "KP solitons, higher Bruhat and Tamari orders". Associahedra, Tamari lattices and related structures. Ed. by Folkert Mülller-Hoissen, Jean Marcel Pallo, and Jim Stasheff. Vol. 299. Progress in Mathematical Physics. Tamari memorial Festschrift. Birkhäuser/Springer, Basel, 2012, pp. 391–423.

[2] Yuji Kodama and Lauren Williams. "KP solitons and total positivity for the Grassmannian". Invent. Math. 198.3 (2014), pp. 637–699.