# Newton's Method in the p-adic World: Density and Root Bias

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## 1 Introduction

Have you ever used Newton's method, one of the most familiar iterative algorithms in mathematics? It provides a way of finding solutions to polynomial $f$, by repeatedly improving a guess until it converges to a root. As the derivative of $f$ is nonzero:

$$N_f(x) = x - \frac{f(x)}{f'(x)}$$

In the classical setting of real or complex numbers, it has been a cornerstone of numerical analysis for centuries. Due to this success, one may naturally ask: what happens if we attempt the method within a stranger arithmetic world?

In p-adic world, "distance" is measured in terms of divisibility by prime number. To be more specific, numbers 0 and 81 are quite far away on the real-axis. Indeed, they are considered very close in 3-adic setting, as the difference is divisible by a high power of 3. After introducing the new "distance", the behavior of iterative processes becomes unpredictable and Newton's method is no exception. Instead of the reliable convergence we expect, the p-adic case could fail to converge.

## 2 Objective

Our project set out to explore this unusual behaviour and mainly focused on two closely related iterative algorithms: Newton's Method and McMullen's Method [1] (generalization of Newton).

There are three key procedures to understand; once we are given polynomial $f$, prime bound $X$, maximum iterations and iterative algorithm (initialization):

- For a given starting value $x_0$. After iterations, we get proportion of these primes that converge to a root of $f$;
- For these primes captured, whether they prefer to converge to a certain root rather than another;
- Repeat the previous steps by modifying the initials.

Unlike the real numbers, these methods can fail to converge altogether in the p-adic world. Following the first step, we get what we call density. This allow us to estimate how often convergence happens when varies primes. Moreover, the second phenomenon is root bias: the tendency of the method to favor certain roots over others, almost as if the prime itself has a "preference".

## 3 Progress

To investigate these fascinating observations, we combined theoretical and computational approaches.

On the theoretical side, we followed and extend the techniques provided in a paper of Faber and Voloch [2]. We proved the theorems still hold for McMullen's method. Especially, there exist infinitely many primes for which McMullen method converge and fail to converge.

On the computational side, we implemented large-scale experiments in Mathematica. We ran across thousands of primes for hundreds of starting points. This allowed us to estimate the relative frequency with which the iteration converges to different roots and to test whether the apparent bias persists as the prime grows larger.

## 4 Results

The infinitely many primes of convergence and non-convergence, do shift our focus from 'if' to 'how'. This means there can never be a universal guarantee of success: the outcome depends delicately on the arithmetic of the prime itself.

Our computations could confirm this in a more intuitive way. Our first observation is that for many families of maps we studied, densities have comparable patterns. Furthermore, our findings suggest that the root bias is genuine at small prime ranges, but appears to diminish when larger primes are considered. However, with our current computations, it remains an open question whether this bias persists for all and what underlying cause it.

In summary, our project illustrates how transplanting a classical numerical method into the p-adic setting reveals surprising new behaviours. What is routine and reliable over the reals becomes subtle, irregular, and arithmetic-dependent over the p-adics. All these have aroused my further interest in number theory and dynamical systems. For example, can these density patterns be explained by deeper number-theoretic invariants, or connected to entropy notions in dynamical systems? These questions remain open and continue to inspire my interest in further research.

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## References

[1] Curt McMullen. "Families of rational maps and iterative root-finding algorithms". In: *Annals of Mathematics* (1987), pp. 467–493.

[2] José Felipe Voloch Xander Faber. "On the Number of Places of Convergence for Newton's Method over Number Fields". In: *Journal de Théorie des Nombres de Bordeaux* (2010).