p-adic Dynamics and Failure of Newton's Method (Part 1)

Holly Krieger, Jingyi Le[1], Annalaura Pegoraro, Ethan Sosin

DPMMS, University of Cambridge
[1] Cambridge Mathematics Open Internship

Acknowledge & Opening

- Today: Newton's method in the p-adic setting
  paper of Faber-Voloch + McMullen maps
- Key Question:
  For which primes p does Newton iteration converge to a root?
  How often does the convergence occur as we vary over primes?

Background – Newton's Method

Definition

Newton's method is a root-finding algorithm, where $f' \neq 0$

$$x_{n+1} = N_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

Goal: approximate the roots of given polynomials

Background – p-adic

Definition

p-adic valuation is defined as given a nonzero rational $x = p^k \frac{m}{n}$, where m,n coprime to p. Then,

$$ord_p(x) = ord_p(p^k \frac{m}{n}) = k \tag{2}$$

ie. the power of p

We'll use $ord_p(x)$ to define p-adically convergence in $\mathbb{Q}_p$

p-adic Absolute Value

Definition

p-adic Absolute Value of x is defined as

$$|x|_p = p^{-ord_p(x)} \tag{3}$$

Definition

$\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value.

Background – p-adic

Definition

Given a root α of f, the newton sequence $(x_n)_{n\geq 1}$ **p-adically convergence** in $\mathbb{Q}_p$ if:

$$|x_n - \alpha|_p = p^{-ord_p(x_n - \alpha)} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (4)$$

ie. the difference is divisible by higher and higher powers of p

for example, take sequence $x_n = 2^n$, $\alpha = 0$ and $p = 2$ we get

sequence: 2,4,8,...

difference: $2, 2^2, 2^3, \ldots$

Background – Bad Prime

However, a problem arises: what if

$$f'(x) \equiv 0 \bmod p$$

Recall Newton's method:

$$N_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

Background – Bad Primes

Definition

Bad primes is a finite set of primes for N_f if (mod p):

- $\mathit{Resultant}(f, f') \equiv 0$ (shared root)

Note: $f'(x) \equiv 0 \bmod p$ suggest the problem, but it is insufficient

For these bad primes, Newton is not well-defined, so exclude.

FV – root bias

Given $f(x) = x^3 - x$, roots = $\{1, 0, -1\}$

- $N_f(x) = \frac{2x^3}{3x^2-1}$;
- $Res[2x^3, 3x^2 - 1] = -4$, bad reduction $p = 2$

Classified the primes into good primes and bad primes

FV – Good Primes

- take good prime $p = 5$ as an example
- each simple root mod 5 uniquely lifts to a root in $\mathbb{Q}_5$ via Hensel's lemma. So newton iteration always converge to correspond lifted roots in $\mathbb{Q}_5$

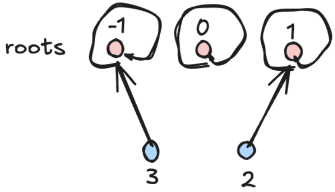

Figure: Newton map mod p=5, can lift to $\mathbb{Q}_5$ later

FV – Bad Primes

Recall $N_f(x) = \frac{2x^3}{3x^2-1}$

- take bad prime $p = 2$, check by hand
- take $x_0 = 3 : x_1 = \frac{27}{13}$ as an example;
  $ord_2(x_1 - (-1)) = ord_2(\frac{40}{13}) = 3$ (num contain 2^3)
  $ord_2(x_1 - 0) = ord_2(\frac{27}{13}) = 0$ (num contain 2^0)
  $ord_2(x_1 - 1) = ord_2(\frac{14}{13}) = 1$ (num contain 2^1)
  so $x_0 = 3$ converges to root -1

FV – Bad Primes (continue)

- continue calculate to get 4 different starting point result here
- $x_0 = \{2, 4\} \rightarrow \textit{root}\ 0$; $x_0 = \{5\} \rightarrow \textit{root}\ 1$; $x_0 = \{3\} \rightarrow \textit{root}\ -1$

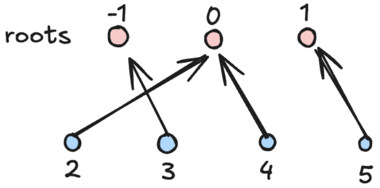

Figure: Newton map in $\mathbb{Q}_2$

FV – root bias observed

Observations:

- for good primes $p = 5$, $x_0 = 2$ converges to root 1
- for bad primes $p = 2$, $x_0 = 2$ converges to root 0

An idea came out:

for a given starting point, vary with numerous primes p,
whether the points prefer convergence to certain root?

PIRFA with GC

Definition

A **purely iterative algorithm** is rational map between space of polynomials of degree d and the space of rational functions of degree k:

$$T : Poly_d(\simeq \mathbb{C}^d) \to Rat_k(\simeq \mathbf{P}^{2k+1}) \quad (5)$$

Definition

We say a map T is **purely iterative root finding algorithm which is generally convergent** if on an open dense $\mathcal{U} \subset Poly_d$, we have $\forall f \in \mathcal{U}$, T_f^{on} converges to a root of f $\forall z \in V \subset \mathbb{C}$, V open dense

Superconvergent

Definition

T_f is **Superconvergent** if its critical points fixed by itself and concide with the roots of f; or equivalently,

$$\forall \alpha \; s.t \; f(\alpha) = 0, \; f'(\alpha) = 0 \; and \; T_f(\alpha) = \alpha \tag{6}$$

Examples of generally convergent algorithms

- $d = 2$, Newton is unique degree 2 superconvengent algorithm
  In fact, Newton does not converge for cubic or higher degree
- $d = 3$, McMullen is unique degree 4 superconvergent algorithm
- $d \geq 4$, no such algorithm exists

McMullen – superconvergent root-finding algorithm

Definition

McMullen maps are defined via a modified Newton-type process: given a cubic polynomial $f(x) = x^3 + ax + b$ and $h(x) = 3ax^2 + 9bx - a^2$, one constructs the rational function

$$q(x) = \frac{f(x)}{h(x)}$$

and defines the associated map $T_f(x)$ as the Newton map of $q(x)$. That is,

$$T_f(x) = x - \frac{(x^3 + ax + b)(3ax^2 + 9bx - a^2)}{3ax^4 + 18bx^3 - 6a^2x^2 - 6abx - 9b^2 - a^3}. \qquad (7)$$

McMullen (continue)

- new rational function of degree 4 if $\Delta(f) = -4a^3 - 27b^2 \neq 0$, otherwise it's a constant map;
- fixed points: roots of f and roots of h;
- cubic-convergence near the simple root, faster than Newton's

McMullen – main theorems

might seem contradictory at first:

Theorem (infinite many convergence)

For a root α of f, the sequence (x_n) converges p-adically to α for infinitely many primes p of $\mathbb{Q}$.

Theorem (infinite many non-convergence)

The sequence (x_n) fails to converge in $\mathbb{P}^1(\mathbb{Q}_p)$ for infinitely many primes p of $\mathbb{Q}$.

In summary, convergence and divergence each occur for infinitely many primes

if $\rightarrow$ how

How does the convergence behave as we consider larger and larger primes?

Part 1 end

Thank you!