Primordial Gravitational Waves from Cosmic Strings and Inflation

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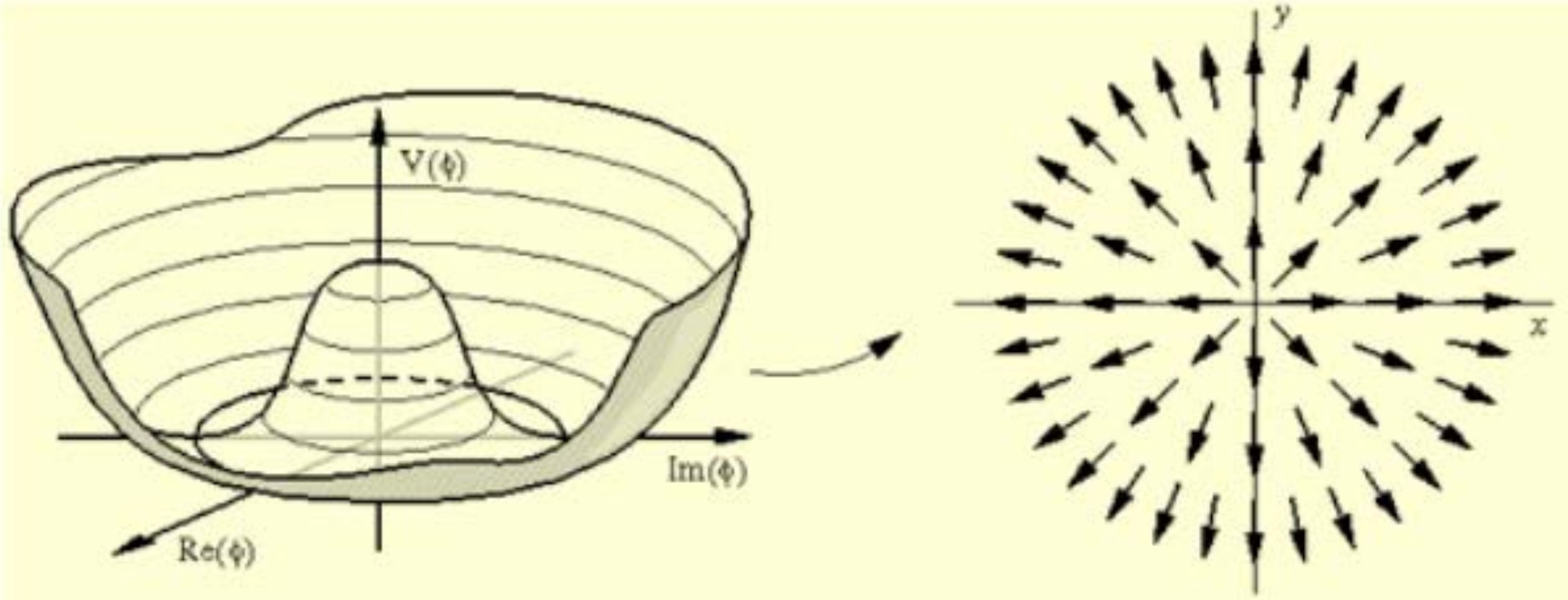
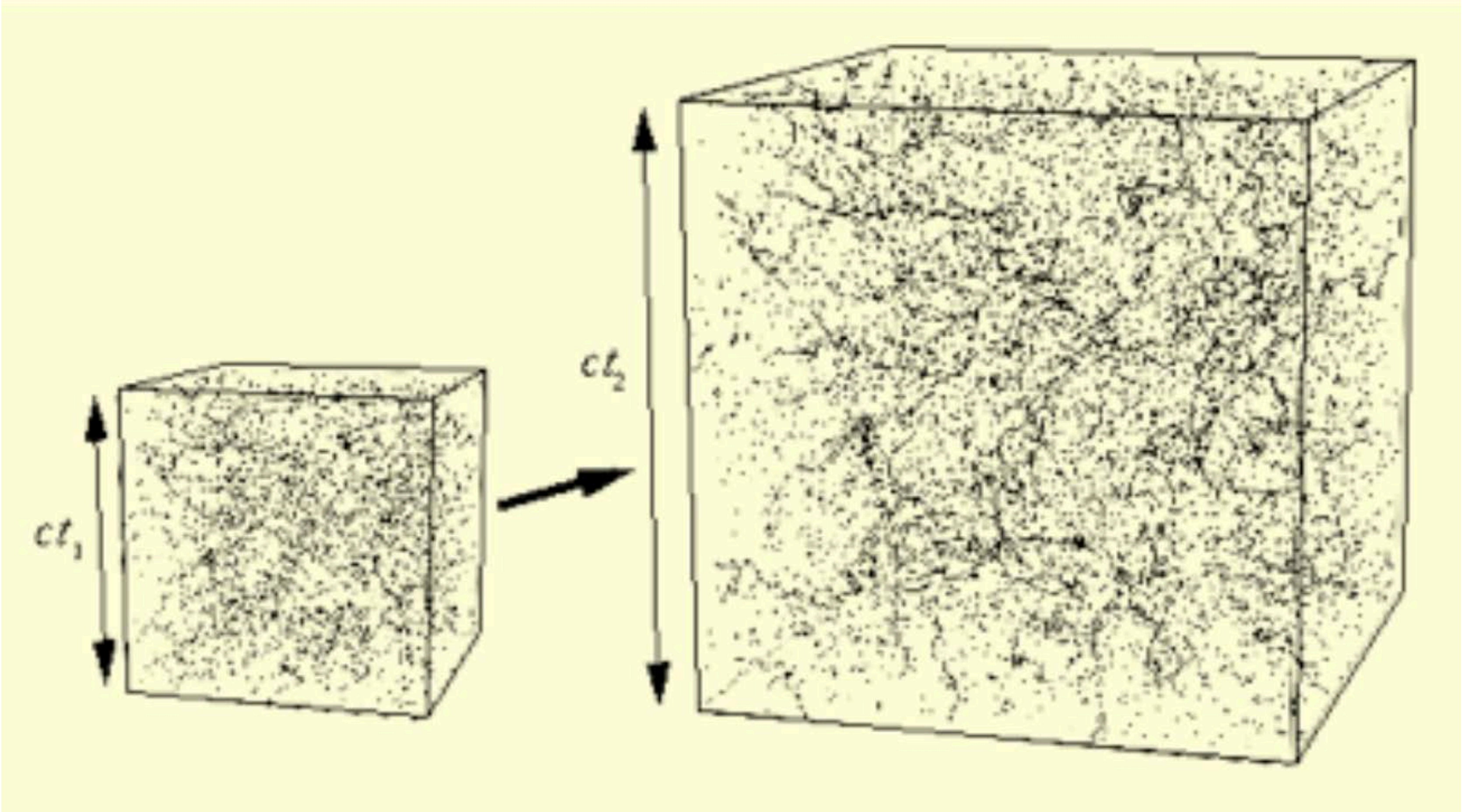
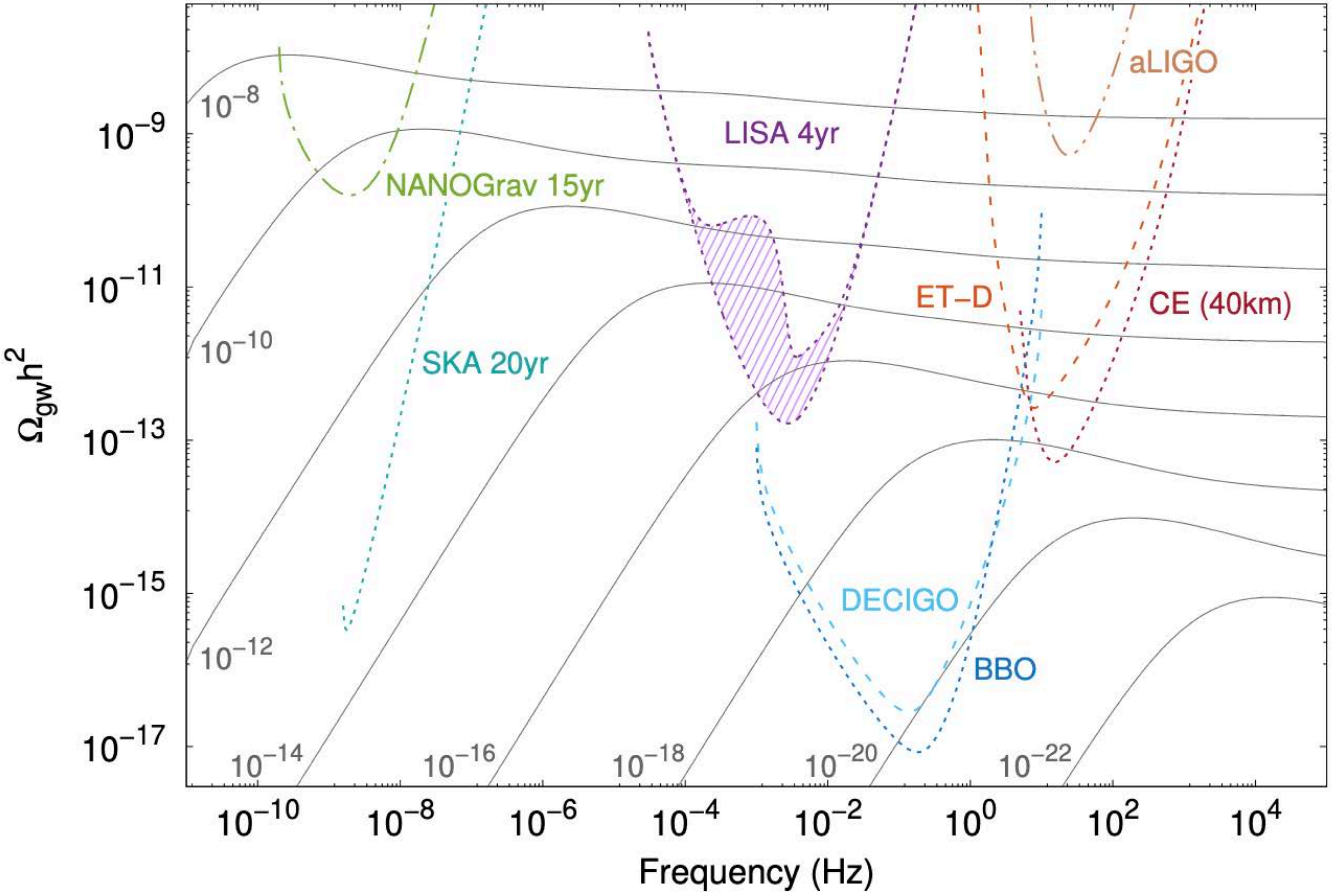
PART 1: Gravitational waves from cosmic strings

INTRODUCTION:

- Universe cooled, symmetry-breaking phase transitions occurred -> (Kibble mechanism) -> Topological defects
- Cosmic strings are infinitesimally thin and extremely dense one-dimensional objects which form when an axial or cylindrical symmetry is broken.
- Evolution of cosmic string network -> cosmological expansion, intercommuting -> GW radiation
- Current goals: probing their SGWB across PTA-LIGO-LISA bands, pushing sensitivity to string tensions below current limits.







Wachter, Olum & Blanco-Pillado, arXiv:2411.16590 (2024)

COSMIC STRING NETWORK EVOLUTION

On large scales, one can describe statistically the cosmic string evolution through the characteristic length scale L and the Root-Mean-Squared (RMS) velocity $\bar{v}$ - Velocity dependent One-Scale (VOS) model:

$$\frac{d\bar{v}}{dt} = (1-\bar{v}^2)\left[\frac{k(\bar{v})}{L} - 2H\bar{v}\right]$$

$$\frac{dL}{dt} = (1+\bar{v}^2)HL + \frac{\tilde{c}}{2}\bar{v}$$

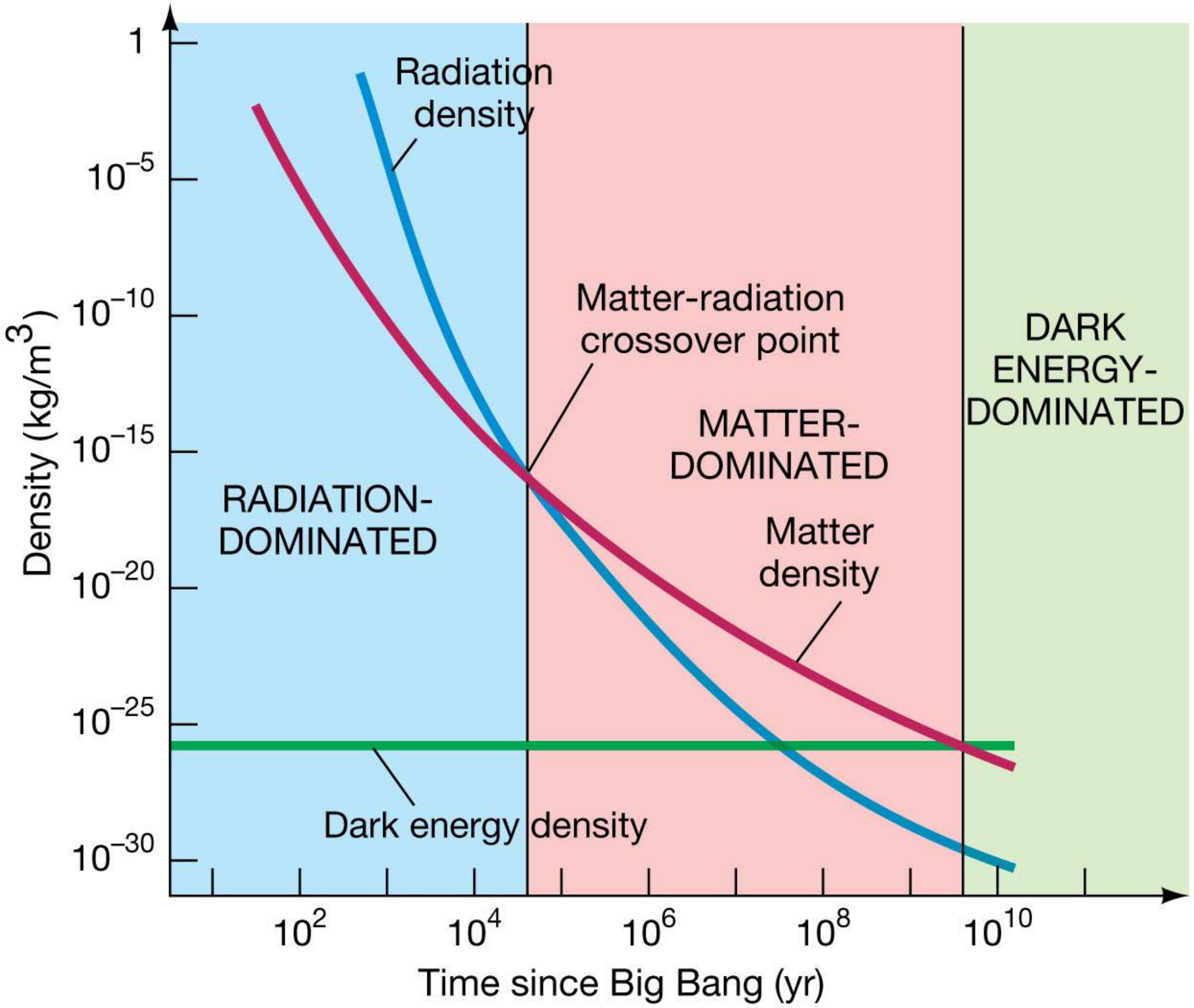
$$\tilde{c} = 0.23 \pm 0.04$$

Where $H \equiv \frac{\dot{a}}{a}$, And $k(\bar{v}) = \frac{2\sqrt{2}}{\pi}(1-\bar{v}^2)(1+2\sqrt{2}\bar{v}^3)\frac{1-8\bar{v}^6}{1+8\bar{v}^6}$

Attractor solution (Linear scaling regime): $a \propto t^\beta$ $\quad 0 < \beta < 1$

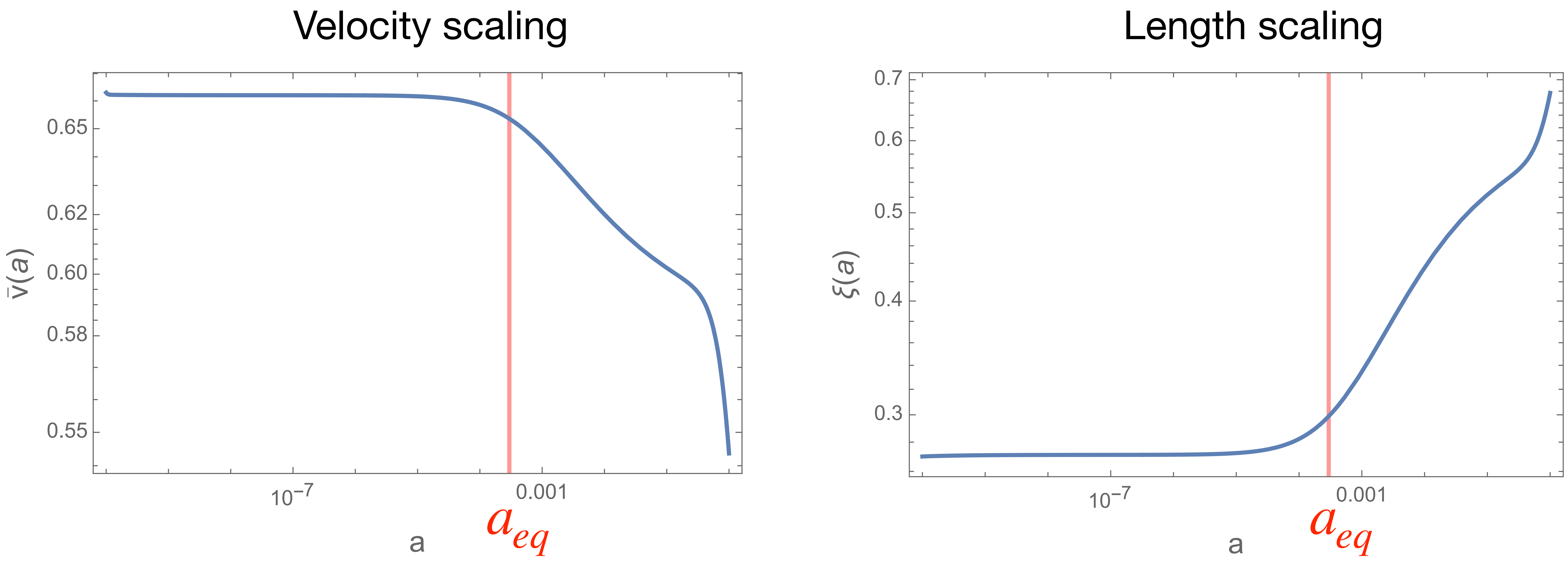
$$L = \xi_\beta t \qquad \frac{d\bar{v}_\beta}{dt} = 0$$

Where $\xi_\beta^2 = \frac{k(k+\tilde{c})}{4\beta(1-\beta)}$ $\quad \bar{v}_\beta^2 = \frac{k}{k+\tilde{c}}\frac{1-\beta}{\beta}$



General background evolution:

$$H(a) = H_0\sqrt{\Omega_r(a_0/a(t))^4 + \Omega_m(a_0/a(t))^3 + \Omega_\Lambda}$$



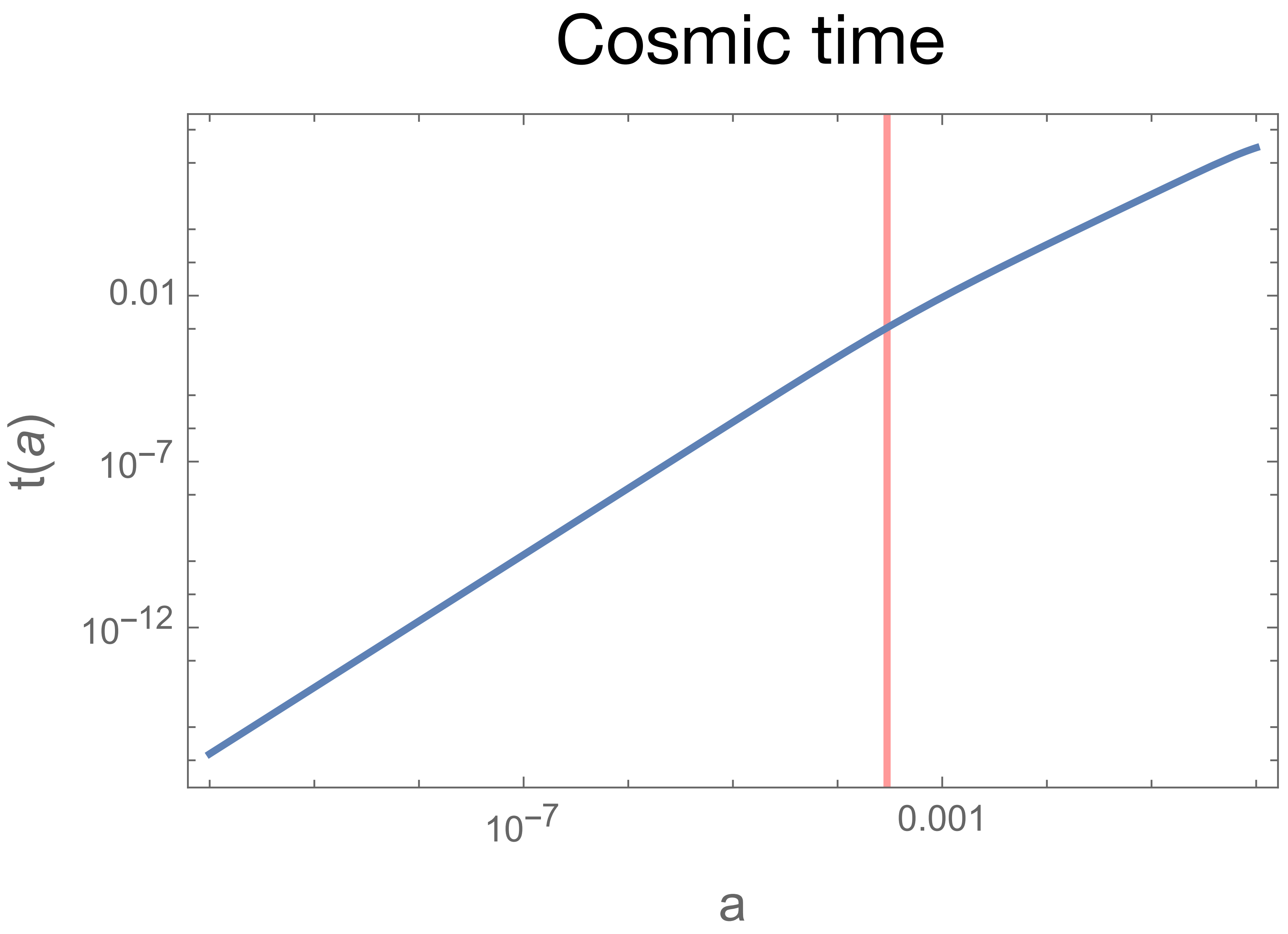
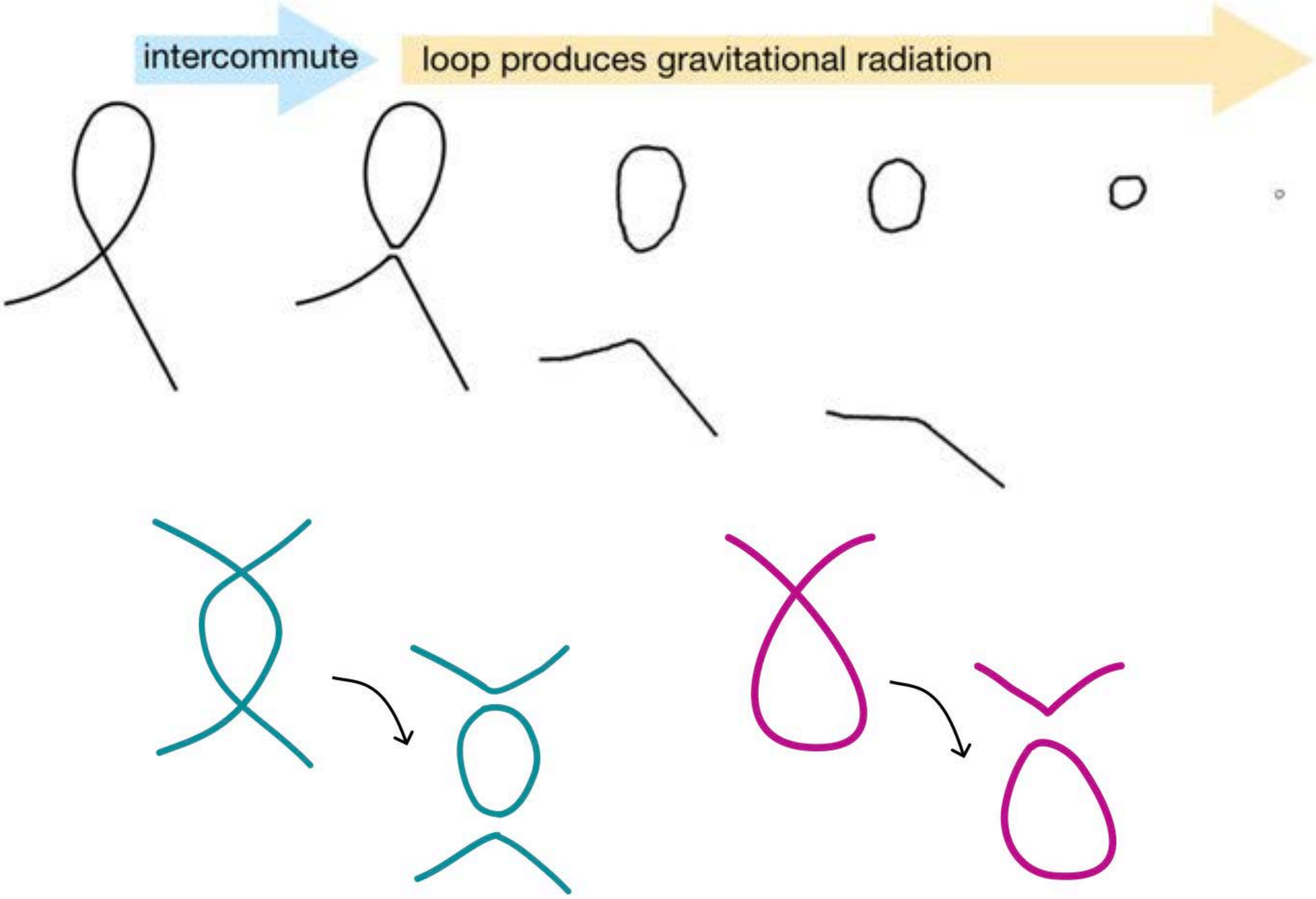


FIG. 1: Evolution of ξ, the RMS velocity $\bar{v}$ of a cosmic string network as well as cosmic time as a function of scale factor.


intercommute
loop produces gravitational radiation

STOCHASTIC GRAVITATIONAL WAVE BACKGROUND

Loops emit GWs in a discrete set of frequencies: $$f_j = \frac{2j}{l}$$

Power emitted into each harmonic mode:

$$\frac{dE_{gw,loop}}{dt} = P_j G\mu^2$$ Where $$P_j = \frac{\Gamma}{\varepsilon} j^{-q}$$

Spectral index

$$q = \frac{4}{3},\ \frac{5}{3},\ 2$$

Cusps ($\frac{4}{3}$), Kinks ($\frac{5}{3}$), Kink-kink collisions (2)

Normalisation factor: $$\varepsilon = \sum_{m=1}^{\infty} m^{-q}$$ → $$\sum_j P_j = \Gamma$$ → $$\Gamma \sim 50$$

Total power: $$P = \sum_j \frac{dE_j}{dt} = G\mu^2 \sum_j P_j = \Gamma G\mu^2$$

STOCHASTIC GRAVITATIONAL WAVE BACKGROUND

Amplitude of the SGWB generated by cosmic string loops - spectral energy of GW:

$$\Omega_{gw}(f) = \frac{1}{\rho_c}\frac{d\rho_{gw}}{d\log f}$$

Where $\rho_c = 3H_0^2/8\pi G$

$$\frac{d\rho_{gw}}{df}(t) = 2\pi \int_{t_i}^{t_0} dt' \left(\frac{a(t')}{a(t)}\right)^3 \int_0^{l(t')} l\, dl\, n(l,t') g\left(\frac{a_0}{a(t')}2\pi f l\right) G\mu^2$$

Scale factor today: $a_0 = a(t_0)$

Redshift factors: $f = f_{emit}\frac{a(t')}{a(t)}$

Function $g(x)$ is normalised by: $\int_0^\infty g(x)dx = \Gamma$

Model a discrete emission spectrum: $g(z) = \sum_j P_j\, \delta(z - 4\pi j)$ Where $z = (a_0/a(t'))2\pi f l$

STOCHASTIC GRAVITATIONAL WAVE BACKGROUND

In terms of the power spectrum: Number of modes considered

$$\Omega_{gw}(f) = \sum_{j=1}^{n_*} P_j \, \Omega_{gw}^j(f)$$

where

$$\Omega_{gw}^j(f) = \frac{16\pi}{3}\left(\frac{G\mu}{H_0}\right)^2 \frac{\Gamma}{f} \int_{ti}^{t_0} jn(l_j(t'), t') \left(\frac{a(t')}{a_0}\right)^5 dt'$$

is the contribution of the j-harmonic mode of emission to the SGWB.

$l_j(t') = 2ja(t')/fa_0$ is the physical length of the loops that radiate in the j-harmonic mode at time t'.

Useful relation: $\Omega_{gw}^j(jf) = \Omega_{gw}^1$

LOOP PRODUCTION FUNCTION

Energy lost into loops: $\left.\frac{d\rho}{dt}\right|_{\text{loops}} = \tilde{c}\bar{v}\frac{\rho}{L}$

Energy density

$\rho = \mu/L^2$

$$\tilde{c}\bar{v}\frac{\rho}{L} = \mu\int_0^\infty l f(l,t)dl$$

Number of loops with length between l and $l + dl$ produced per unit time

Loops shrink due to gravitational radiation at constant rate: $\frac{dl}{dt} = -\Gamma G\mu$ → $l(t) = l_b - \Gamma G\mu(t - t_b)$

$$l_b \equiv l(t_b) = \alpha L(t_b)$$

$\alpha < 1$

Loop size parameter

Loop production function: $$f(l,t) = \frac{\tilde{c}\bar{v}(t)}{\sqrt{2}\alpha L(t)^4}\delta(l - \alpha L)$$

NUMBER DENSITY

Total number density:

$$n(l,t) = \int_{t_i}^{t} dt_b f(l_b, t_b) \left(\frac{a(t_b)}{a(t)}\right)^3$$

Substituting in loop production function:

$$n(l,t) = \sum_i \left\{ \left(\alpha \frac{dL}{dt}\Big|_{t=t_b^{(i)}} + \Gamma G\mu\right)^{-1} \frac{\tilde{c}\bar{v}(t_b^{(i)})}{\sqrt{2}\alpha L(t_b^{(i)})^4} \left(\frac{a(t_b^{(i)})}{a(t)}\right)^3 \right\}$$

Scaling regime:

$$t_b^{(i)} = \frac{l + \Gamma G\mu t}{\alpha\xi_\beta + \Gamma G\mu} \qquad a \propto t^\beta \qquad 0 < \beta < 1$$

$$n(l,t) = \frac{\tilde{c}v_\beta}{\sqrt{2}\alpha\xi_\beta^4 t^{3\beta}} \frac{(\alpha\xi_\beta + \Gamma G\mu)^{3-3\beta}}{(l + \Gamma G\mu t)^{4-3\beta}}$$

Radiation: $\beta = 1/2$

Matter: $\beta = 2/3$

ANALYTICAL APPROXIMATION

Sousa, Avelino & Guedes, *Phys. Rev. D* **101** (2020) 103508.

Produced and decay in the radiation era:

$$\Omega^r_{gw} = \frac{128}{9}\pi A_r \Omega_r \frac{G\mu}{\epsilon_r}\left[\left(\frac{f(1+\epsilon_r)}{B_r\Omega_m/\Omega_r + f}\right)^{3/2} - \left(\frac{f(\epsilon_r+1)}{B_i + f}\right)^{3/2}\right] \quad \epsilon_r = \frac{\alpha\xi_r}{\Gamma G\mu} \quad A_r = \frac{\tilde{c}}{\sqrt{2}}\frac{\bar{v}_r}{\xi_r^3} \quad B_i = \frac{2}{\Gamma}\sqrt{\frac{2H_0\Omega^{1/2}}{t_{pl}(\epsilon_r+1)}}$$

$$B_r = \frac{4H_0\Omega_r^{1/2}}{\Gamma G\mu}$$

Produced in the radiation era and decay in the matter era:

$$\Omega^{rm}_{gw}(f) = 32\sqrt{3}\pi(\Omega_m\Omega_r)^{3/4}H_0\frac{A_r}{\Gamma}\frac{(\epsilon_r+1)^{3/2}}{f^{1/2}\epsilon_r}\left\{\frac{(\Omega_m/\Omega_r)^{1/4}}{(B_m(\Omega_m/\Omega_r)^{1/2}+f)^{1/2}}\left[2+\frac{f}{B_m(\Omega_m/\Omega_r)^{1/2}+f}\right] - \frac{1}{(B_m+f)^{1/2}}\left[2+\frac{f}{B_m+f}\right]\right\}$$

Produced and decay in the matter era:

$$\epsilon_m = \frac{\alpha\xi_m}{\Gamma G\mu} \quad A_m = \frac{\tilde{c}}{\sqrt{2}}\frac{\bar{v}_m}{\xi_m^3} \quad B_m = \frac{3H_0\Omega_m^{1/2}}{\Gamma G\mu}$$

$$\Omega^m_{gw}(f) = 54\pi H_0\Omega_m^{3/2}\frac{A_m}{\Gamma}\frac{\epsilon_m+1}{\epsilon_m}\frac{B_m}{f}\left\{\frac{2B_m+f}{B_m(B_m+f)} - \frac{1}{f}\frac{2\epsilon_m+1}{\epsilon_m(\epsilon_m+1)} + \frac{2}{f}log\left(\frac{\epsilon_m+a}{\epsilon_m}\frac{B_m}{B_m+f}\right)\right\}$$

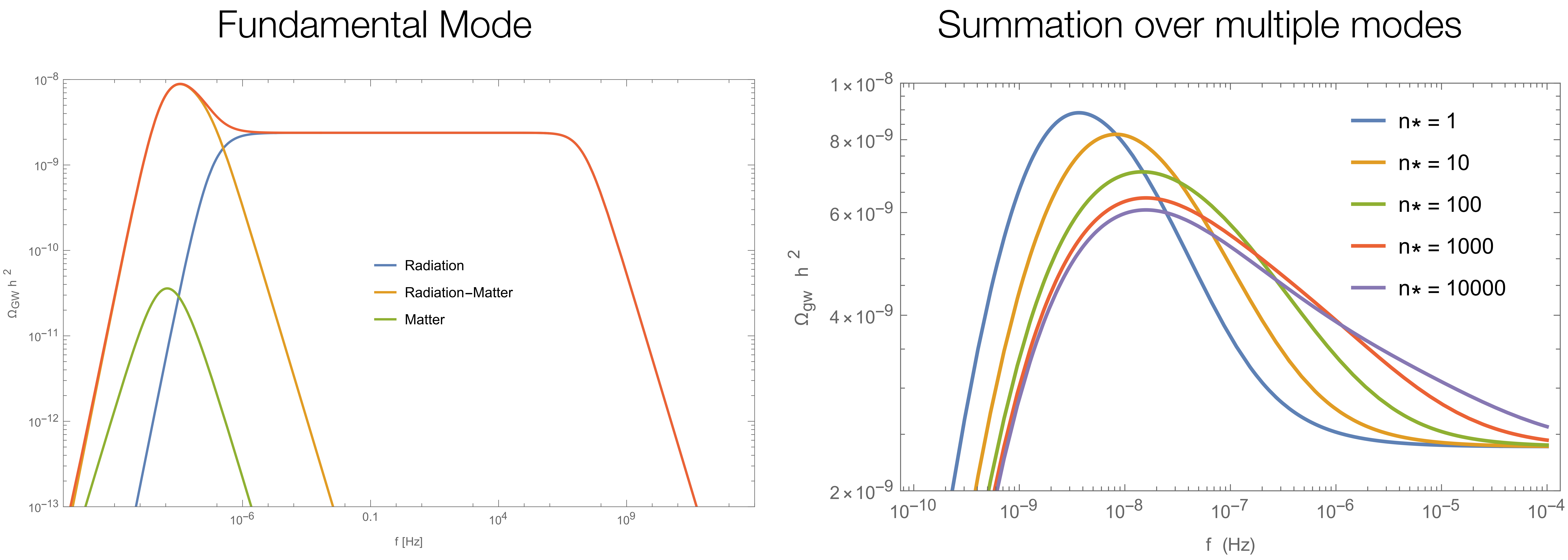


FIG. 2: Analytical approximation to the stochastic gravitational wave background generated by cosmic string networks with $G\mu = 10^{-10}$ and $\alpha = 0.1$.

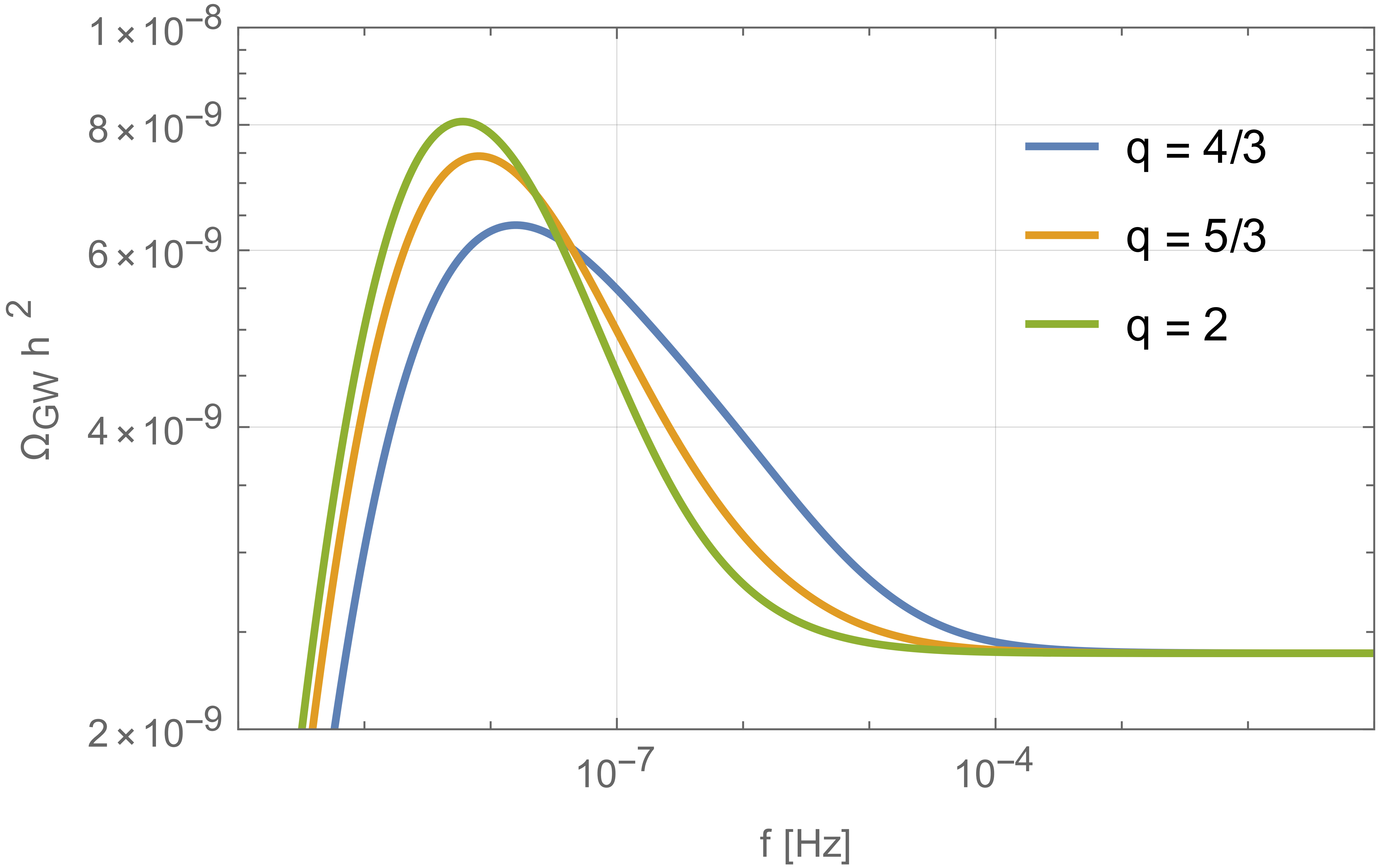
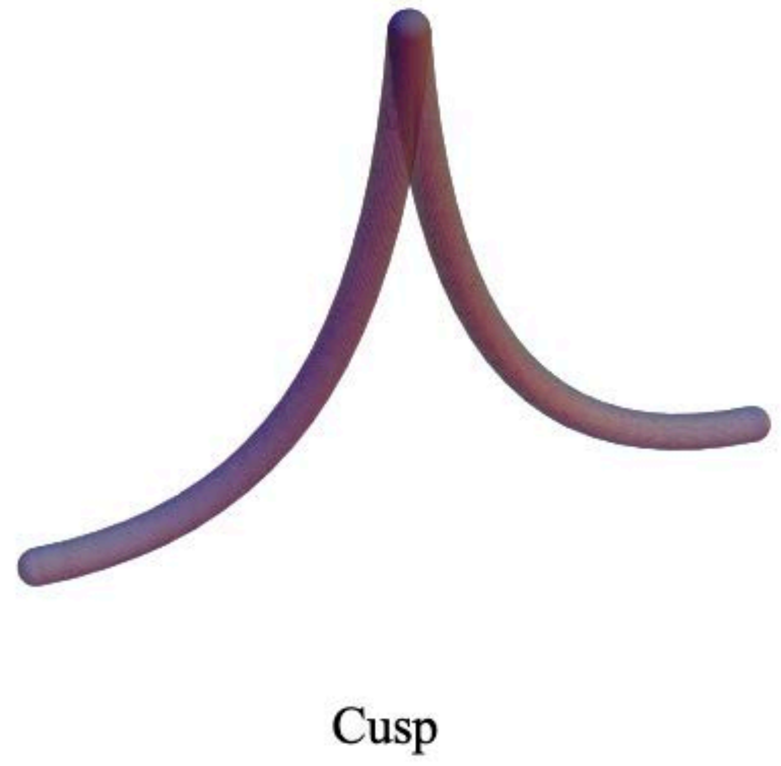
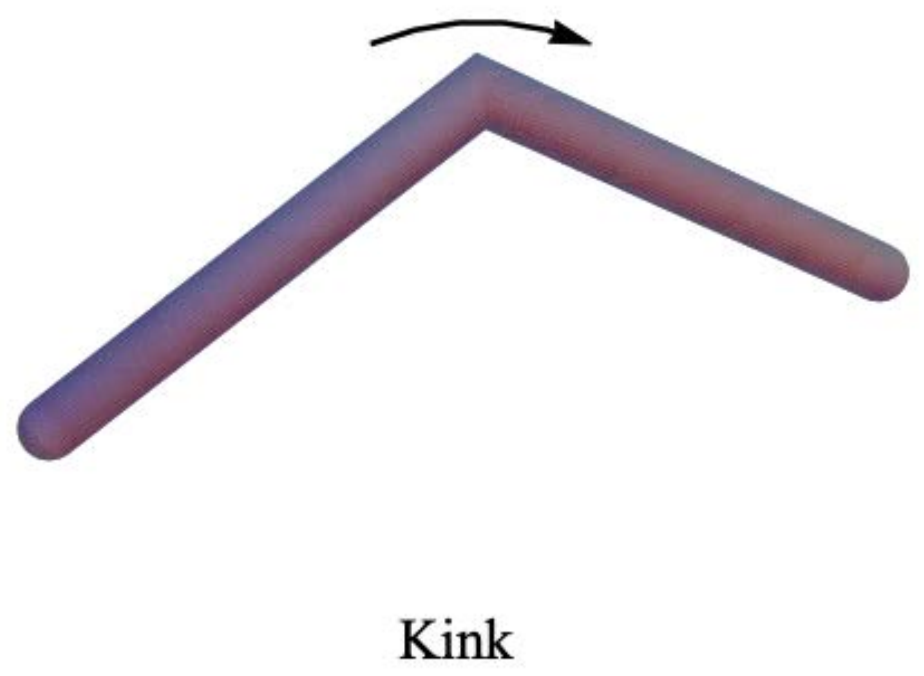


FIG. 3: Analytical approximation to the stochastic gravitational wave background generated by cosmic string networks with $G\mu = 10^{-10}$, $\alpha = 0.1$ and different spectral indices ($n* = 10^3$).

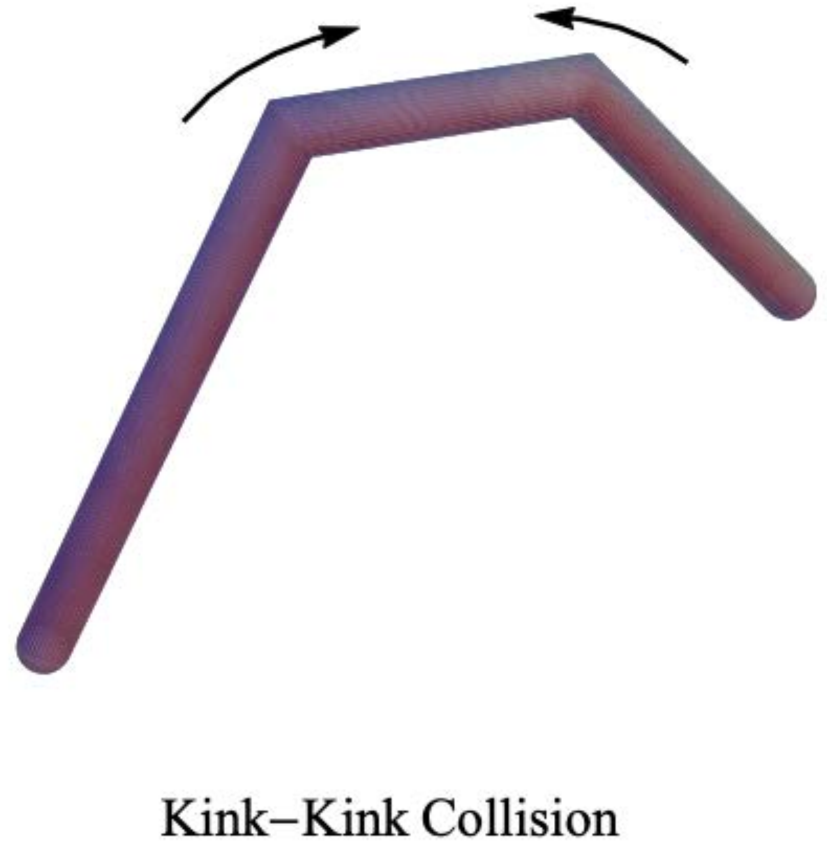
Three main mechanisms for loop GW emission:



$q = 4/3$



$q = 5/3$



$q = 2$

Long, Hyde & Vachaspati, *JCAP* **09** (2014) 030.

NUMERICAL SCHEME

Comoving number density: $N \equiv a^3 n$

$$\partial_t N + \partial_l(\dot{l}N) = a^3 f(l,t)$$

Use methods of characteristics $\dot{l} = u(l,t) = -\Gamma G\mu$

Along each such curve PDE collapses to an ODE: $\frac{dN}{dt} = a^3 f(l,t)$

$$N(l,t) = \int_{t_i}^{t} a(\tau)^3 \frac{\tilde{c}\bar{v}(\tau)}{\sqrt{2}L(\tau)^4} \delta(l + \Gamma G\mu(t-\tau) - \alpha L(\tau)) d\tau$$

Define: $g(t) = \alpha L(t) + \Gamma G\mu t$ → $t_b = g^{-1}(l + \Gamma G\mu t)$ → $N(l,t) = \frac{a(t_b)^3 \tilde{c}\bar{v}(t_b)}{\sqrt{2}\alpha L(t_b)^4 |dg(t_b)/dt|}$

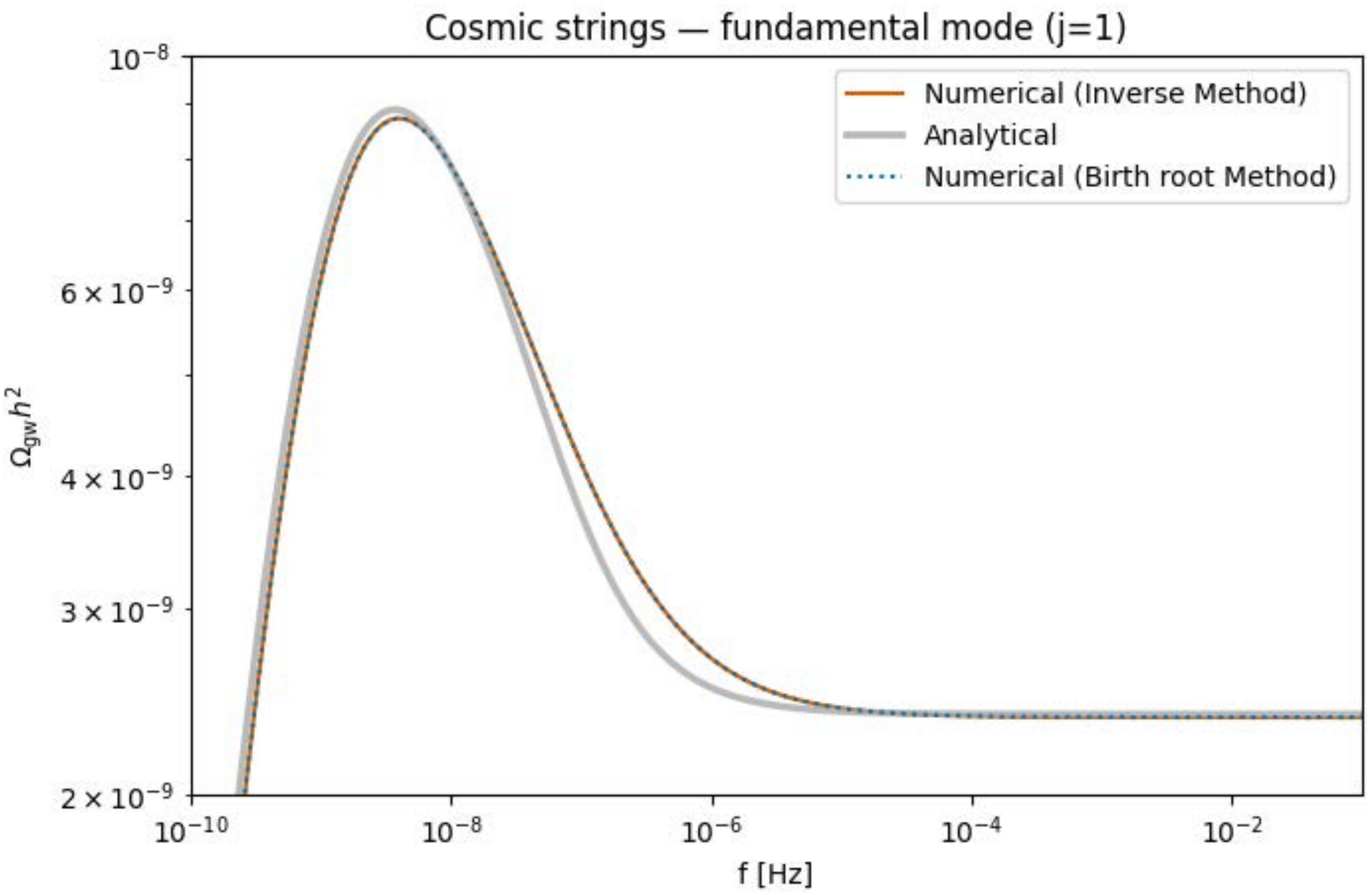
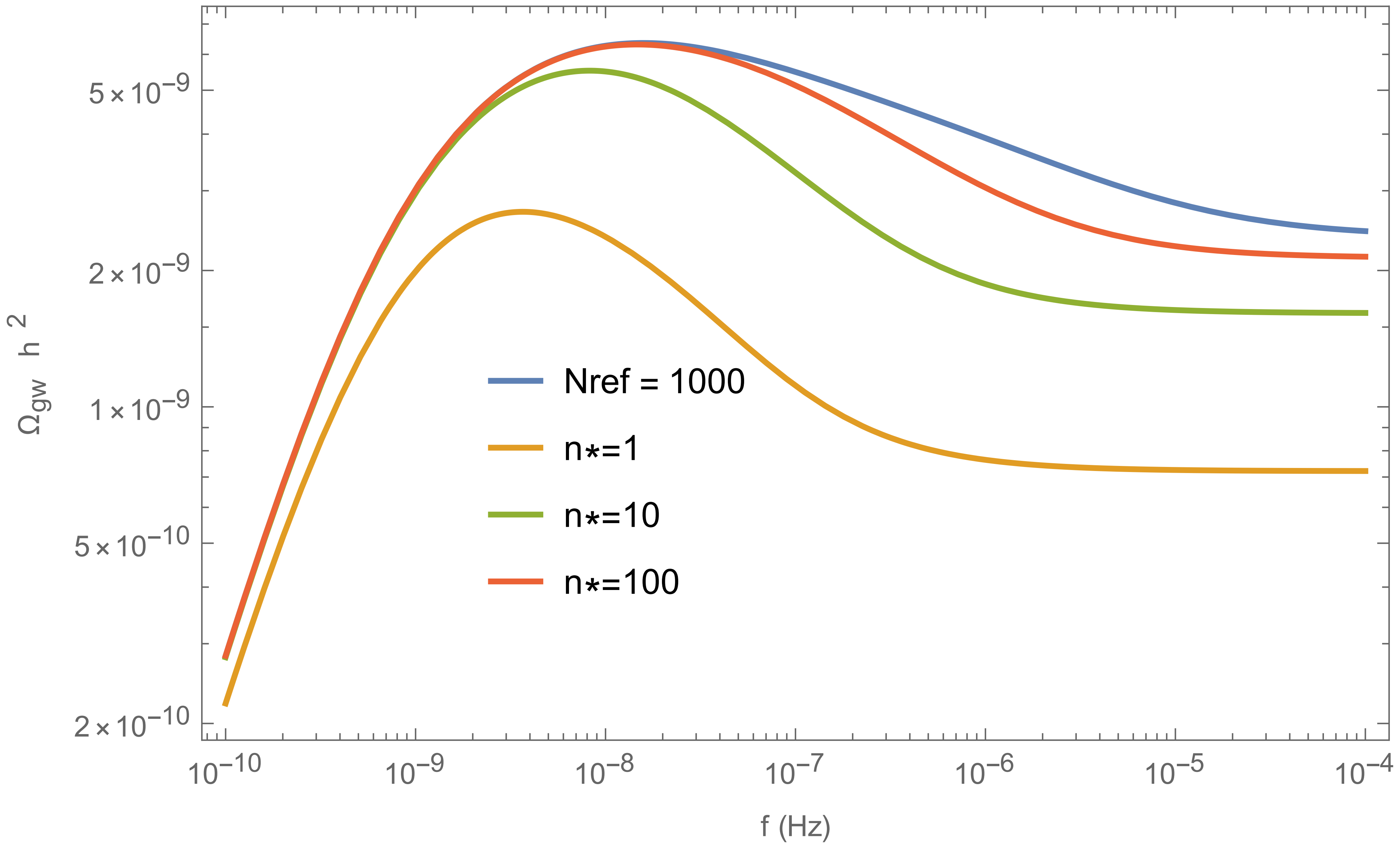


FIG. 4: Comparison between analytical approximation to the stochastic gravitational wave background and the SGWB obtained numerically ($G\mu = 10^{-10}$ and $\alpha = 0.1$).

FUTURE RESEARCH

- Explore other types of loop production functions (non-discrete) as well as extending to scenarios where loop production does not occur at a single lengthscale.

- Extend Lasse Gerblich's research on the effects of gravitational back reaction on cosmic string networks and see how the back reaction affects the SGWB spectrum.



PART 2: Gravitational Waves from Inflation

COSMOLOGICAL PERTURBATION THEORY

Full metric of GR decoupled into its background and perturbation parts: $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$

$$ds^2 = -\bar{N}^2(1+2A)dt^2 + 2a\bar{N}B_i dt dx^i + a^2(\delta_{ij} + 2E_{ij})dx^i dx^j$$

Time slicing

$$\bar{N} = a$$

$$B_i = \partial_i B + \hat{B}_i$$

$$E_{ij} = C\delta_{ij} + \partial_{\langle i}\partial_{j\rangle}E + \partial_{(i}\hat{E}_{j)} + \hat{E}_{ij}$$

where $\partial_{\langle i}\partial_{j\rangle}E = \left(\partial_i\partial_j - \frac{1}{3}\nabla^2\delta_{ij}\right)E$

Scalar perturbations: (A, B, C, E)

Tensor perturbation: $\hat{E}_{ij} = h_{ij}/2$

Vector perturbations: $(\hat{E}_i, \hat{B}_i)$

GW produced by quantum effects during inflation

MAGNETIC PART OF WEYL TENSOR

Weyl curvature tensor C_{abcd} may be split into electric and magnetic parts:

$$E_{\mu\nu} = C_{\mu\nu\rho\sigma}u^\rho u^\sigma$$

$$B_{\mu\nu} = \frac{1}{2}\eta_{\mu\rho\sigma\alpha}C^{\rho\sigma}{}_{\nu\beta}u^\beta u^\alpha$$

First order: $B^{(1)}_{ij} = \varepsilon_{kl(i}\partial^k h'^{(1)l}_{j)}$

Second order: $B^{(2)}_{ij} = \varepsilon_{kl(i}\partial^k H'_{j)}{}^{(2)l}$

Consists of second order $h^{(2)}_{ij}$ and quadratic first order scalar terms

PLAN:

Extract B_{ij} from numerical relativity code simulations of inflation

→

Convert to Fourier Space and invert the 'curl'

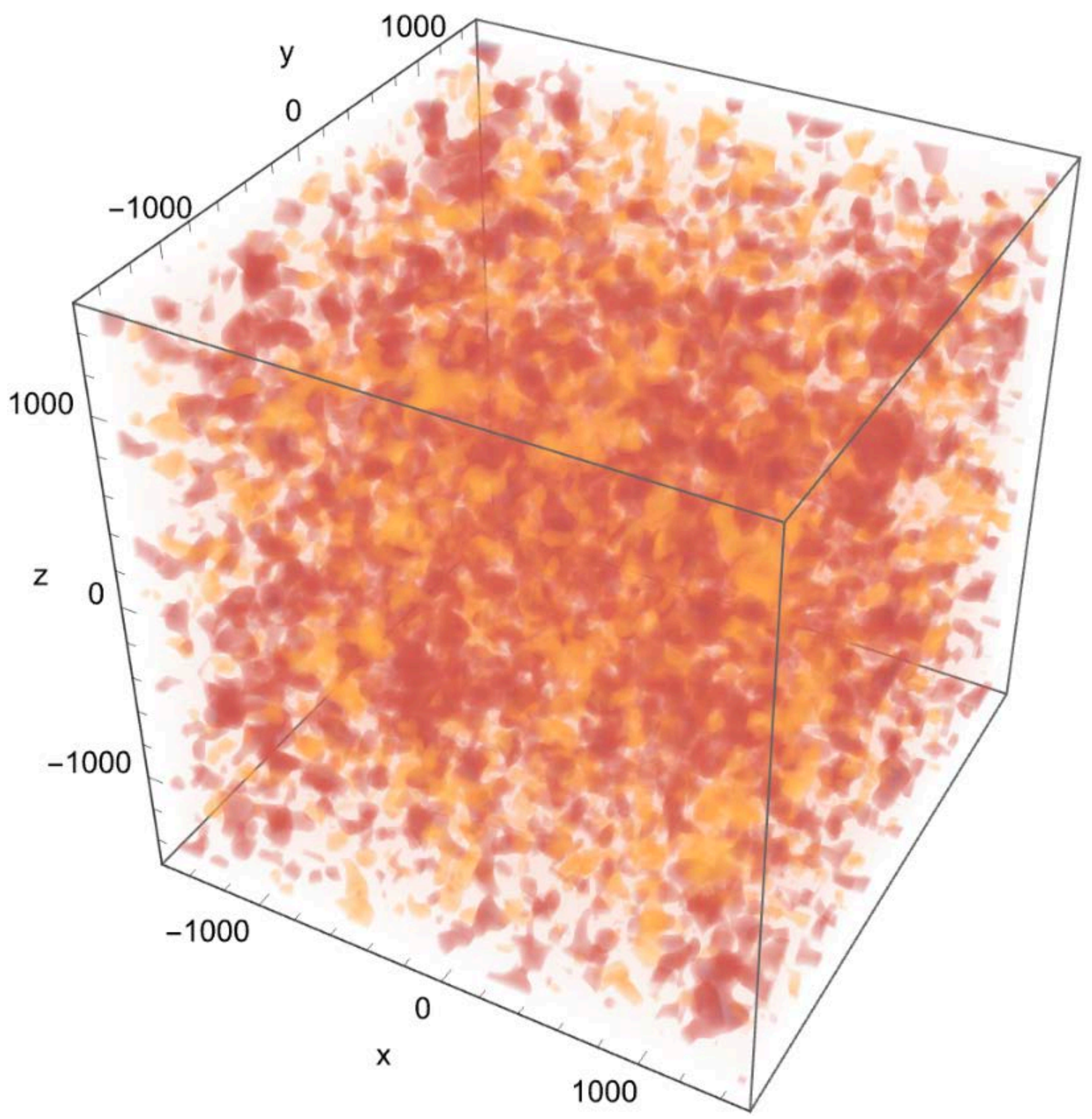
→

Obtain tensor perturbation $H_{ij} = h_{ij}^{(1)} + H_{ij}^{(2)}$

↓

Compare with Ericka's results to validate $h_{ij}^{(1)}$ and extend to scalar modes $H_{ij}^{(2)}$

- Consider other second order in scalar perturbations gauge invariant variables such as the Cotton York Tensor.
- Establish the physical observable/ gravitational wave strain being measured.



Thank you