Philippa Fawcett Internship Programme 2025

p-adic Dynamics and the Failure of Newton's Method

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Finding the roots of a polynomial (those special values where the polynomial equals zero) is a classic problem in Mathematics, with applications in both science and engineering. The most famous root-finding algorithm is *Newton's method*. It's an iterative process: you start from an initial point and generate a sequence by repeatedly applying Newton's map, a function determined by your polynomial. When the generated sequence gets closer and closer to a root, we say the method *converges*. In the familiar world of real numbers, this works very well if your first guess is reasonably close to a root.

But what happens when we change our very definition of 'closeness'? Number theorists are interested in whether Newton's method can find roots in the world of *p-adic numbers*. This number system, which extends the rational numbers, defines 'closeness' in a unique way. Instead of being measured by distance on a number line, it's all about divisibility by a prime number p.
In the p-adic world, a number is considered 'small' if it's divisible by a high power of p. This leads to some bizarre outcomes. For example, in the 2-adic world, the sequence $2, 4, 8, 16, 32, \ldots$ is actually converging to 0, because each number is divisible by a higher power of 2.

A classic result, *Hensel's Lemma*, tells us when Newton's method works well for a given prime p. But what if you look at *all* primes at once? Researchers Faber and Voloch proved that, for a given polynomial, Newton's method fails to p-adically converge for infinitely many primes, and conjectured it does so for 100% of them. Later, applying *Chebotarev's Density Theorem*, Faber and Towsley proved that the method fails for a strictly positive density of primes.

Interested in these phenomena, we extended these results to a specific class of root-finding algorithms with two members: Newton's and *McMullen's methods*. Both methods are examples of *dynamical systems*, which are objects generated by the repeated application of a mathematical function. We therefore wondered if Newton's and McMullen's maps shared a common dynamical feature that could explain these results.
The answer is yes: we discovered the underlying reason is that they both exhibit *superattracting behaviour*. On the real line, this means there are special points that behave like 'mathematical black holes': nearby starting values are pulled in with incredible speed. This led us to generalise the results to a much wider class of maps that share this property.
To explore this further, we developed a computational tool in *Mathematica* to study the p-adic convergence of these methods. Collecting extensive data from different examples led us to formulate a new and stronger conjecture.

Finally, what I loved about this project was seeing the interplay between computational experiments, Complex Dynamics, and Number Theory. It was an opportunity to observe how ideas from various areas of Mathematics can come together to form a complicated yet harmonious framework.