p-adic Dynamics and the Failure of Newton's Method
(Part 2)

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Background

Given a polynomial f, recall the Newton map $N_f(x) = x - \frac{f(x)}{f'(x)}$. Then, Newton's sequence (x_n) is defined by $x_{n+1} = N_f(x_n)$, $x_0 \in \mathbb{Q}$.

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Theorem (Faber-Voloch)
Newton's sequence (x_n) converges p-adically to a root of f for infinitely many primes p, and fails to p-adically converge for infinitely many primes p.

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Definition
The *natural lower density* of a set of primes $\mathcal{S}$ is

$$\delta(\mathcal{S}) = \liminf_{X\to\infty} \frac{\#\{p \in \mathcal{S} \mid p \leq X\}}{\#\{p \leq X\}}$$

Density of primes yielding convergence

Conjecture (Faber-Voloch)
Let $C(\mathbb{Q}, f, x_0)$ be the set of primes for which (x_n) converges p-adically to a root of f. Then the natural density of the set $C(\mathbb{Q}, f, x_0)$ is zero.

For instance, take the polynomial $f(x) = x^3 - 1$ and a starting point x_0. We can study the behaviour of

$$\delta(X) = \frac{\#\{p \leq X \mid (x_n) \text{ converges to a root of } f \text{ in } \mathbb{Q}_p\}}{\#\{p \leq X\}}.$$

Density of primes yielding convergence

$$f(x) = x^3 - 1, \quad x_0 = 2$$

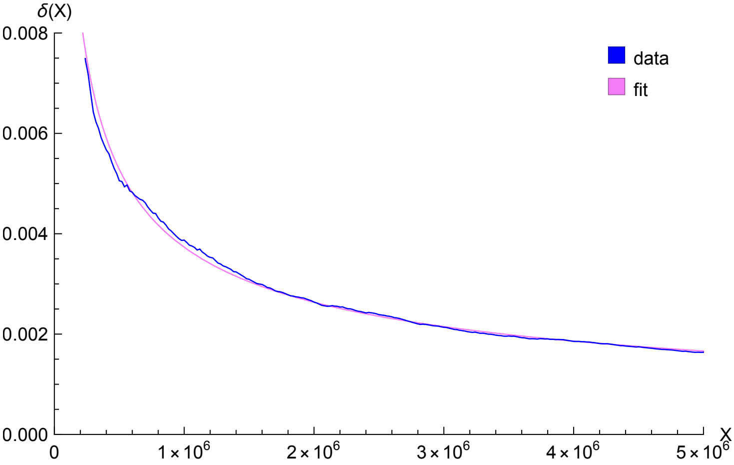

Figure: Plot of $\delta(X)$ for X up to 5,000,000.

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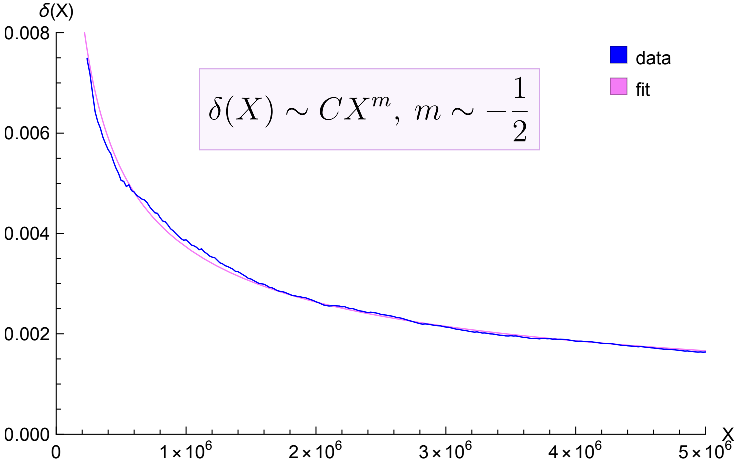

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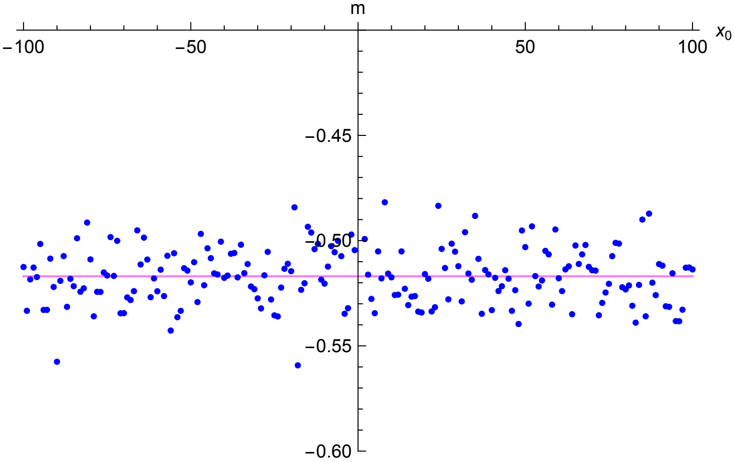
$$f(x) = x^3 - 1, \quad \delta(X) \sim CX^m$$



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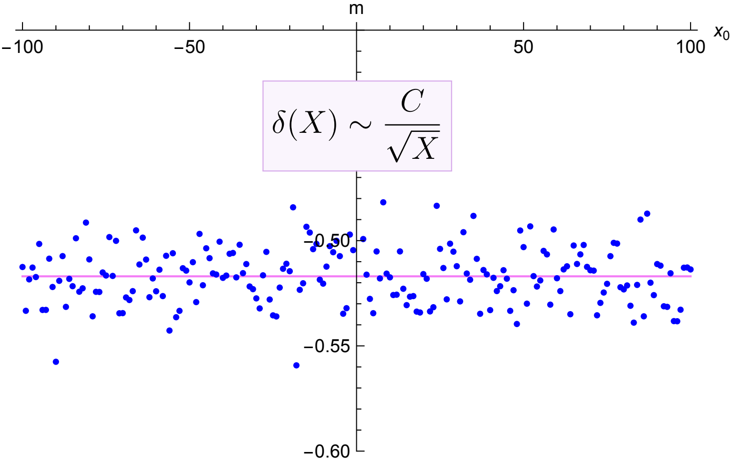


$$\delta(X) \sim \frac{C}{\sqrt{X}}$$

Figure: Plot of m for different values of x_0.

Root bias

Faber and Voloch also studied the polynomial $g(x) = x^3 - x$, which has roots $\{-1, 0, 1\}$.

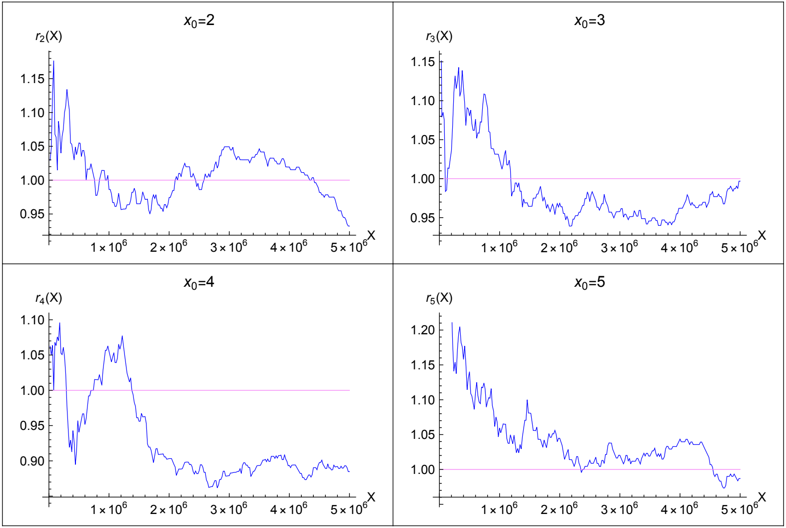
Question Does Newton's sequence converge p-adically more often to $+1$ or -1?

The data they collected seemed to suggest a bias towards the root $+1$.

We now study the behaviour of the ratio

$$r(X) = \frac{\#\{p \leq X \mid x_n \to +1 \text{ in } \mathbb{Q}_p\}}{\#\{p \leq X \mid x_n \to -1 \text{ in } \mathbb{Q}_p\}}.$$

Root bias



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A note on how to test for convergence/divergence

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Theorem (Faber-Voloch)
After discarding finitely many primes p, Newton's sequence (x_n) converges p-adically to a root of f if and only if $f(x_n) \equiv 0 \mod p$ for some n.

After discarding finitely many primes, the sequence $(x_n \mod p)$ is well defined and eventually periodic.

A note on how to test for convergence/divergence

Let's consider $f(x) = x^3 - 1$, $x_0 = 2$.

Then, $N_f(x) = \dfrac{2x^3 + 1}{3x^2}$, $x_{n+1} = N_f(x_n)$.

For all the following primes the only root of f modulo p is $\alpha = 1$, so to satisfy the condition $f(x_n) \equiv 0 \pmod{p}$, it is sufficient to check if the reduced sequence (x_n mod p) hits 1.

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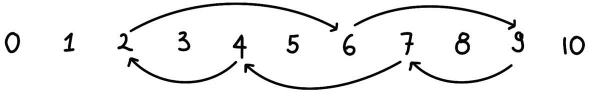
$\underline{p = 11}$



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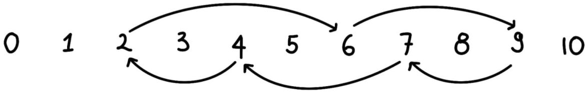

$p = 17$ $x_0 = 2$, $x_1 \equiv 0 \pmod{17}$, x_2 will have a power of 17 in the denominator and so will all subsequent iterates

Back to the theory

As a consequence of Chebotarev's Denisty Theorem, the following theorem holds.

Theorem (Faber-Towsley)
Newton's sequence (x_n) fails to p-adically converge for a set of primes p with positive lower density.

Question Does this hold also for McMullen's map?

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Theorem
McMullen's sequence (x_n) fails to p-adically converge for a set of primes p with positive lower density.

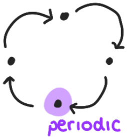
A *rational map* is a quotient

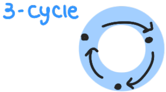
$$T(z) = \frac{a_d z^d + \ldots + a_0}{b_d z^d + \ldots + b_0} \in \mathbb{C}(z)$$

with no common factor and a_d, b_d not both zero.
The *orbit* of a point $\alpha \in \mathbb{C} \cup \{\infty\}$ is the sequence

$$\mathcal{O}(\alpha) = \{T(\alpha), T^2(\alpha), T^3(\alpha), \ldots\}$$

Definition
A point α is *periodic* if $T^n(\alpha) = \alpha$ for some n. The smallest such n is called *period*. If α is periodic of period n, then we say $\mathcal{O}(\alpha) = \{\alpha, T(\alpha), \ldots, T^{n-1}(\alpha)\}$ is an *n-cycle*.





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Definition
An n-cycle $\mathcal{O}(\alpha) = \{\alpha, T(\alpha), ..., T^{n-1}(\alpha)\}$ is *superattracting* if $(T^n)'(\alpha) = 0$.

Rational maps...

...with superattracting behaviour

Theorem
Given a rational map T with a superattracting n-cycle there are infinitely many primes p for which the sequence (x_n) defined by $x_{n+1} = T(x_n)$ converges to the cycle.

Theorem
Given a rational map T with superattracting n-cycles, the sequence (x_n) defined by $x_{n+1} = T(x_n)$ fails to p-adically converge for a set of primes p with positive lower density.

Thank you!