Functional inequalities for quantum Rényi divergences

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We start by recalling two important notions from quantum mechanics. A density matrix (or density operator) ρ is an operator acting on a Hilbert space $\mathcal{H}$ such that ρ is positive semidefinite and $\mathrm{Tr}(\rho) = 1$. In physics, the pure state of a quantum system can be described by a vector $|\Psi\rangle$ (the so-called ket notation describing a state that lives in a Hilbert space). Equivalently, we can describe the state of the system by its density matrix $\rho = |\Psi\rangle \langle\Psi|$.

The formal definition of a quantum channel reads: a quantum channel is a completely positive, trace-preserving (CPTP) linear map $\mathcal{E} : B(\mathcal{H}_{in}) \to B(\mathcal{H}_{out})$ where $B(\mathcal{H})$ represents the set of bounded operators acting on the Hilbert space $\mathcal{H}$, and $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ are the initial and final Hilbert spaces of our system. Quantum channels are just linear operators that take as an input a density matrix and return another density matrix. Some important examples: quantum channels describing the time evolution of a system (we will study these), partial trace, etc.

A quantum divergence is a functional $D(\_||\_)$ that takes as an input two density matrices ρ and σ, and returns a real number. It quantifies how distinct two density matrices are. All divergences obey the Data Processing Inequality (DPI): $D(\rho||\sigma) \geq D(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$ for any quantum channel $\mathcal{E}$. A quantum divergence becomes zero when there is no distinguishability between the inputs, i.e. $D(\rho||\sigma) = 0$ if and only if $\rho = \sigma$.

We begin by providing some important examples of quantum divergences, namely the Relative entropy and the Belavkin-Staszewski relative entropy (or BS relative entropy). The relative entropy, also known as the Umegaki relative entropy, is the most studied and widely used quantum divergence. It is given by:

$$D(\rho||\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)] \tag{1}$$

The Belakvin-Staszewski relative entropy is lesser known, but equally interesting, and it is given by:

$$\hat{D}(\rho||\sigma) = \mathrm{Tr}[\rho \log(\rho^{\frac{1}{2}}\sigma^{-1}\rho^{\frac{1}{2}})] \tag{2}$$

Besides the Relative entropy and the BS relative entropy, quantum Rényi divergences represent other important quantities. In this work, we focus on studying the Geometric Rényi divergence. There are also other divergences, such as Sandwiched Rényi divergence and Petz-Rényi divergence, which have already been studied in the literature. The interesting property of the Geometric Rényi divergence in comparison with the previous ones is that in the limit $\alpha \to 1$ it will converge to the Belvkin-Staszewski relative entropy, while the others converge to the relative entropy. The first step in our work is to find a functional inequality that will relate the Geometric Rényi divergence to its time derivative.

We start by defining the Geometric Rényi divergence. For $\alpha \in (1, 2]$, the Geometric Renyi divergence is defined as:

$$\hat{D}_\alpha(\rho||\sigma) := \frac{1}{\alpha - 1}\log Tr(\sigma^{\frac{1}{2}}(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}})^\alpha\sigma^{\frac{1}{2}}) \tag{3}$$

Quantum Markov Semigroups (QMS) are a framework to describe the time evolution of open systems under the assumption that the environment has no memory (Markov approximation). The infinitesimal generator of the QMS is called Liouvillian, or Lindbladian, and it is denoted by $\mathcal{L}$. The log-Sobolev constant allows us to find a lower bound for the decay rate of a given divergence:

$$-\frac{d}{dt}D(\rho_t||\sigma) \geq 2\alpha(\mathcal{L})D(\rho_t||\sigma) \tag{4}$$

The log-Sobolev constant can be defined using the general expression:

$$\alpha(\mathcal{L}) = \inf_\rho \frac{-\frac{d}{dt}D(\rho_t||\sigma)\big|_{t=0}}{2D(\rho||\sigma)} \tag{5}$$

In a similar way we find the equivalent log-Sobolev constant for the Geometric Rényi divergence:

$$\beta_\alpha(\mathcal{L}) = \inf_\rho \frac{-\frac{d}{dt}\hat{D}_\alpha(\rho_t||\sigma)\Big|_{t=0}}{2\hat{D}_\alpha(\rho||\sigma)} \tag{6}$$

This project focused mainly on two tasks: a) find an expression for the log-Sobolev constant for the Geometric Rényi divergence (particularly, we managed to find an expression for the operator derivative $\partial_t D(\rho_t||\sigma)$), and b) try to find a lower bound for the log-Sobolev constant for the BS relative entropy for a specific Liouvillian (still under work).

References

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