Functional inequalities for quantum Rényi divergences

Andreea Silvia Goia

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Scientific coordinator:

Prof. Dr. Ángela Capel Cuevas

Contents

- Introduction
- Motivation
- Results
- Conclusions and outlook

Contents

Introduction – Density matrix operators

Mathematics

Definition: A density matrix (or density operator) ρ is an operator acting on a Hilbert space H such that:

1. ρ is positive semidefinite
2. $Tr(\rho) = 1$

Physics

- In physics, the state of a quantum system can be described by a vector (usually denoted by a so-called *ket* $|\psi\rangle$) that lives in a Hilbert space.
- Another equivalent possibility is to describe the system not by a ket, but by its density matrix

We distinguish two cases:

a) the system is isolated, in which case the density matrix is $\rho = |\psi\rangle\langle\psi|$ -> pure state

b) the system interacts with an environment, $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ -> ensemble, mixed state

We also have: $\rho = \rho^\dagger$ is Hermitian; a state is pure if and only if $\rho^2 = \rho$.

Introduction – Quantum Channels

Formal definition

A *quantum channel* is a completely positive, trace-preserving (CPTP) linear map

$$\mathcal{E} : B(H_{in}) \quad \rightarrow \quad B(H_{out})$$

where $B(H)$ represents the set of bounded operators acting on the Hilbert space H, and H_{in} and H_{out} are the initial and final Hilbert spaces of our system.

Properties of quantum channels if acting on density matrices:

- positive: if $\rho \geq 0$, then also $\mathcal{E}(\rho) \geq 0$;
- Trace-preserving: since $\mathrm{Tr}(\rho) = 1$, so will $\mathrm{Tr}[\mathcal{E}(\rho)] = 1$;
- complete positivity: $\mathcal{E} \otimes I_k$ is positive for all k.

Basic intuition

Quantum channels are just linear operators that take as an input a density matrix and returns another density matrix.

Some important examples: **quantum channels describing the time evolution of a system** (we will study these), partial trace.

Introduction – Quantum divergences

A quantum divergence is a functional $D(\_||\_)$ that takes as an input two density matrices ρ and σ, and returns a real number. It quantifies how distinct two density matrices are.

Axioms:

I. Continuity: $D(\rho||\sigma)$ is continuous in $\rho, \sigma \geq 0$, wherever $\rho \neq 0$ and $\sigma \gg \rho$;

II. Unitary invariance: $D(\rho||\sigma)$ = $D(U\rho U^{\dagger}||U\sigma U^{\dagger})$ for any unitary U;

III. Normalization: $D(1||1/2) = \log 2$;

IV. Order: if $\rho \geq \sigma$ (i.e. $\rho - \sigma \geq 0$), then $D(\rho||\sigma) \geq 0$;

V. Additivity: $D(\rho \otimes \tau||\sigma \otimes \omega) = D(\rho||\sigma) + D(\tau||\omega)$ for all $\rho, \sigma, \tau, \omega \geq 0$ with $\sigma \gg \rho$ and $\omega \gg \tau$.

All divergences obey the Data Processing Inequality (DPI):

$$D(\rho||\sigma) \geq D(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$$

for any quantum channel $\mathcal{E}$.

A quantum divergence become zero when there is no distinguishability between the inputs, i.e. $D(\rho||\sigma) = 0 \iff \rho = \sigma$.

Introduction – some important examples

We begin by providing some important examples of quantum divergences, namely the Relative entropy and the Belavkin-Staszewski relative entropy (or BS relative entropy).

1) The relative entropy, also known as the Umegaki relative entropy, is the most studied and widely used quantum divergence. It is given by:

$$D(\rho||\sigma) \coloneqq Tr[\rho(\log \rho - \log \sigma)]$$

2) The Belakvin-Staszewski relative entropy is lesser known, but equally interesting, and it is given by:

$$\widehat{D}(\rho||\sigma) \coloneqq Tr[\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

Introduction – quantum Rényi divergences

Besides the Relative entropy and the BS relative entropy, quantum Rényi divergences represent other important quantities.

Maximal divergence:

$$D_{max}(\rho||\sigma) := \log(||\sigma^{-1/2}\rho\sigma^{-1/2}||_\infty)$$

Sandwiched Rényi divergence:

$$\widetilde{D}_\alpha(\rho||\sigma) := \frac{1}{\alpha - 1}\log Tr\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right] \qquad \alpha \in [1, \infty)$$

Geometric Rényi divergence:

$$\widehat{D}_\alpha(\rho||\sigma) := \frac{1}{\alpha - 1}\log Tr\left[\sigma^{\frac{1}{2}}\left(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right)^\alpha\sigma^{\frac{1}{2}}\right] \qquad \alpha \in (1,2]$$

Petz-Rényi divergence:

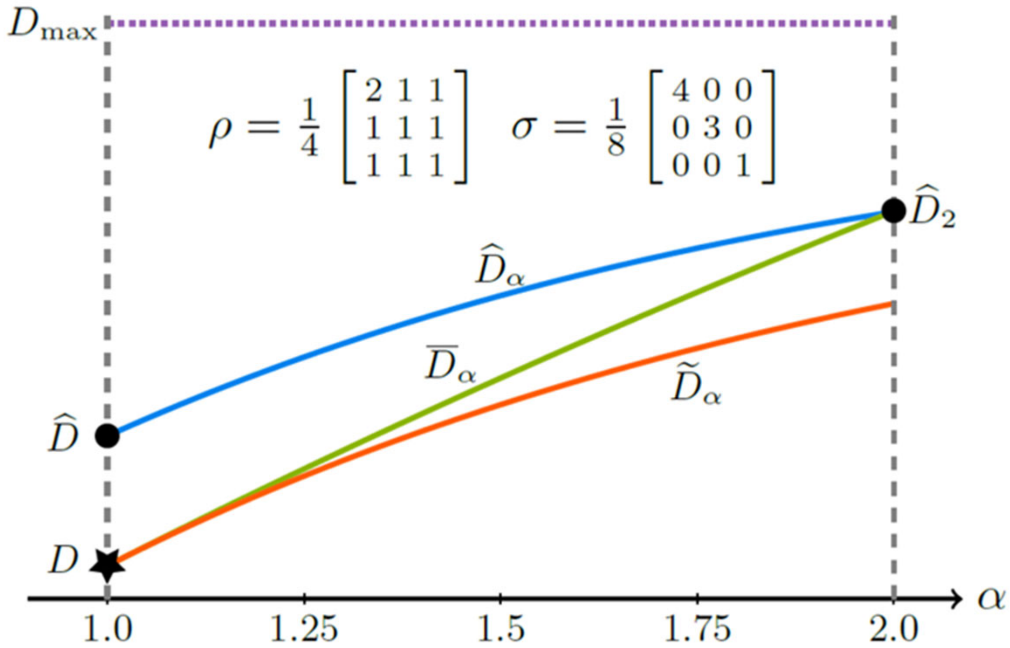
$$\overline{D}_\alpha(\rho||\sigma) := \frac{1}{\alpha - 1}\log Tr[\rho^\alpha\sigma^{1-\alpha}] \qquad \alpha \in (0,1) \cup (1,2)$$

Introduction – important inequalities

And we have two important inequalities for these quantities:

$$D(\rho\|\sigma) \leq \widetilde{D}_\alpha(\rho\|\sigma) \leq \bar{D}_\alpha(\rho\|\sigma) \leq \widehat{D}_\alpha(\rho\|\sigma) \leq D_{\max}(\rho\|\sigma)$$

$$D(\rho_{AB}\|\sigma_{AB}) \leq \widehat{D}(\rho_{AB}\|\sigma_{AB})\,.$$



Contents

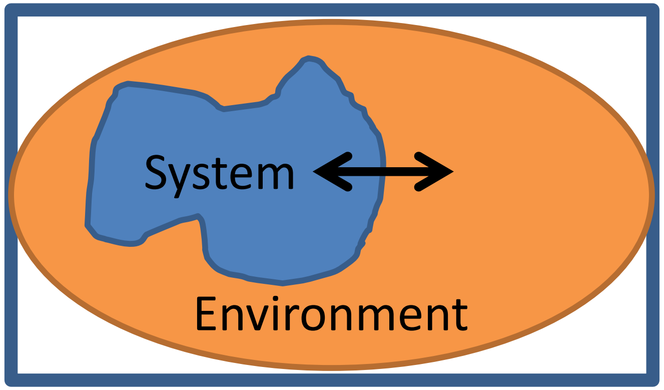
Motivation – open vs closed quantum systems

Isolated system

- Dynamics of closed systems (no interaction with noise or environment)
- Represent an ideal model, but do not exist in reality.
- Evolve reversibly.

Open system

- There are interactions between the system and the environment.
- We can write equations for both the system and the environment.
- The time evolution is not reversible.



Motivation – Quantum Markov Semigroups

Quantum Markov Semigroups (QMS) are a framework to describe the **time evolution** of open systems under the assumption that the environment has no memory (Markov approximation).

$$\mathcal{T}_t^* \circ \mathcal{T}_s^* = \mathcal{T}_{t+s}^* .$$
$$\mathcal{T}_0^* = \mathbb{1} .$$

The infinitesimal generator of the QMS is called **Liouvillian**, or **Lindbladian**, and it is denoted $\mathcal{L}_\Lambda^*$:

$$\frac{d}{dt}\mathcal{T}_t^* = \mathcal{T}_t^* \circ \mathcal{L}_\Lambda^* = \mathcal{L}_\Lambda^* \circ \mathcal{T}_t^* .$$

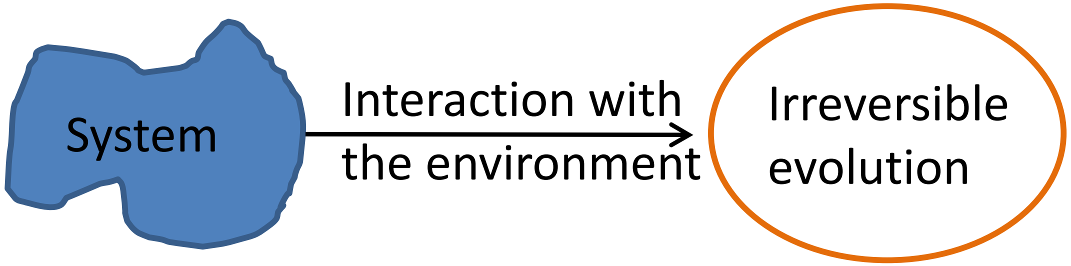
$$\mathcal{T}_t^* = e^{t\mathcal{L}_\Lambda^*} \Leftrightarrow \mathcal{L}_\Lambda^* = \frac{d}{dt}\mathcal{T}_t^* \,|_{t=0} .$$

Motivation – time evolution

We assume that $\{\mathcal{T}_t^*\}_{t\geq 0}$ has a unique full-rank invariant state, which we denote by σ.

Notation: $\rho_t := \mathcal{T}_t^*(\rho)$.

$$\rho_\Lambda \xrightarrow{t} \rho_t := \mathcal{T}_t^*(\rho_\Lambda) = e^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$$



Motivation – log-Sobolev constant

The log-Sobolev constant allows us to find a lower bound for the decay rate of a given divergence:

$$-\frac{d}{dt}D(\rho_t||\sigma) \geq 2\alpha(\mathcal{L})D(\rho_t||\sigma)$$

Integrated, we equivalently have:

$$D(\rho_t||\sigma) \leq D(\rho||\sigma)e^{-2\alpha(\mathcal{L})t}$$

The log-Sobolev constant can be defined using the general expression:

$$\alpha(\mathcal{L}) = \inf_{\rho} \frac{-\frac{d}{dt}D(\rho_t||\sigma)\Big|_{t=0}}{2D(\rho||\sigma)}$$

Contents

Results for the geometric Rényi divergence

Definition For $\alpha \in (1, 2]$, the Geometric Renyi divergence is defined as:

$$\hat{D}_\alpha(\rho||\sigma) := \frac{1}{\alpha - 1} \log Tr(\sigma^{\frac{1}{2}}(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}})^\alpha \sigma^{\frac{1}{2}})$$

- Why is it this quantity interesting?

Proposition. When $\alpha \to 1$ the Geometric Rényi divergence becomes the Belavkin-Staszewski entropy:

$$\lim_{\alpha \to 1} \hat{D}_\alpha(\rho||\sigma) = \hat{D}(\rho||\sigma)$$

Results for the geometric Rényi divergence

We are interested in finding the time derivative:

Proposition. The time derivative of the Geometric Rényi divergence can be written as:

$$\frac{d}{dt}\hat{D}_\alpha(\rho_t||\sigma) = \frac{1}{\alpha - 1}\frac{Tr(\sigma^{\frac{1}{2}}\dot{Z}(t)\sigma^{\frac{1}{2}})}{Tr(\sigma^{\frac{1}{2}}(\sigma^{-\frac{1}{2}}\rho_t\sigma^{-\frac{1}{2}})^\alpha\sigma^{\frac{1}{2}})}$$

where the matrix $\dot{Z}(t)$ is defined in a basis in which $S(t) = \sigma^{-\frac{1}{2}}\rho_t\sigma^{-\frac{1}{2}}$ is diagonal as:

$$[\dot{Z}(t)]_{ij} = \begin{cases} \alpha[S(t)]_{ii}^{\alpha-1}[\dot{S}(t)]_{ii} & \text{for } i = j \\ \frac{[S(t)]_{ii}^\alpha - [S(t)]_{jj}^\alpha}{[S(t)]_{ii} - [S(t)]_{jj}}[\dot{S}(t)]_{ij} & \text{for } i \neq j \end{cases}$$

Remark: This is the only form in which the derivative can be written. The expression cannot be simplified anymore.

Results for the geometric Rényi divergence

Below we detail a bit why this derivative is not trivial.

$$\widehat{D}_\alpha(\rho_t||\sigma) = \frac{1}{\alpha-1}\log Tr\left[\sigma^{\frac{1}{2}}\left(\sigma^{-\frac{1}{2}}\rho_t\sigma^{-\frac{1}{2}}\right)^{\alpha}\sigma^{\frac{1}{2}}\right] = \frac{1}{\alpha-1}\log Tr\left[\sigma^{\frac{1}{2}}S(t)^{\alpha}\sigma^{\frac{1}{2}}\right]$$

$$\Rightarrow \frac{d}{dt}\widehat{D}_\alpha(\rho_t||\sigma) = \frac{d}{dt}\left(\frac{1}{\alpha-1}\log Tr\left[\sigma^{\frac{1}{2}}S(t)^{\alpha}\sigma^{\frac{1}{2}}\right]\right) = \frac{1}{\alpha-1}\log Tr\left[\sigma^{\frac{1}{2}}\frac{d}{dt}\left(S(t)^{\alpha}\right)\sigma^{\frac{1}{2}}\right]$$

The derivative $\frac{d}{dt}(S(t)^{\alpha})$ is an operator derivative. In our case, $\frac{d}{dt}S(t) \coloneqq \dot{S}(t)$ does not commute with $S(t)$ since $\dot{S}(t) = \sigma^{-\frac{1}{2}}\mathcal{L}(\rho_t)\sigma^{-\frac{1}{2}}$, i.e. $[S(t),\dot{S}(t)] \neq 0$.

In this case, we cannot simply write $\frac{d}{dt}(S(t)^{\alpha}) \neq \alpha S(t)^{\alpha-1}\dot{S}(t)$, but rather we need to perform a Fréchet derivative, which is exactly what was done before.

Results for the geometric Rényi divergence

We make use of the time derivative:

Definition The quantity $-\frac{d}{dt}\hat{D}_\alpha(\rho_t||\sigma)\Big|_{t=0}$ is called entropy production, and it is denoted as $\mathrm{EP}(\rho)$.

Proposition. In the case of the geometric Rényi divergence, the entropy production is written as:

$$\mathrm{EP}_\alpha(\rho) = -\frac{1}{\alpha-1}\frac{Tr(\sigma^{\frac{1}{2}}\dot{Z}(0)\sigma^{\frac{1}{2}})}{Tr(\sigma^{\frac{1}{2}}(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}})^\alpha\sigma^{\frac{1}{2}})}$$

With the help of the entropy production, we finally define the log-Sobolev constant:

$$\beta_\alpha(\mathcal{L}) = \inf_\rho \frac{\mathrm{EP}_\alpha(\rho)}{2\hat{D}_\alpha(\rho||\sigma)}$$

Results for the geometric Rényi divergence

Q: How do we check that our computations are correct?
A: We could perform some sanity checks, for example the one below.

It is known that when $[\rho, \sigma] = 0$, we have the following:

$$\hat{D}_\alpha(\rho||\sigma) = \tilde{D}_\alpha(\rho||\sigma)$$

$$-\frac{d}{dt}\hat{D}_\alpha(\rho_t||\sigma)\Big|_{t=0} = -\frac{d}{dt}\tilde{D}_\alpha(\rho_t||\sigma)\Big|_{t=0}$$

These check out for our found expressions, so we can be more confident that we did not make mistakes in our computations.

Results for the geometric Rényi divergence

The earlier expression for the log-Sobolev constant is usually very difficult to work with. Thus, we can resume to a particular case to (at least try) compute it.

Particular case: $\sigma = \frac{I}{d}$ and the depolarizing Lindbladian $\mathcal{L}(\rho) = Tr(\rho)\frac{I}{d} - \rho$.

In this case we can only obtain an interval for the possible value of $\beta_\alpha(\mathcal{L})$:

Theorem For $\mathcal{L}(\rho) = Tr[\rho]\frac{I}{d} - \rho$, for $\alpha \in (1, 2]$ we have:

$$\frac{\alpha}{2(2\alpha-1)}\frac{1}{\log(d)} \geq \beta_\alpha(\mathcal{L}) \geq \frac{\alpha}{2(2\alpha-1)}\frac{d^{\frac{2\alpha-1}{\alpha}}-1}{d^{\frac{2\alpha-1}{\alpha}}\log(d)}$$

Note: this is an improvement of the result obtained by Müller-Hermes and França [4].

Results for the BS relative entropy

Recall the definition of the BS relative entropy:

$$\hat{D}(\rho||\sigma) = Tr[\rho \log(\rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}})]$$

We have the following equivalent form:

Lemma The BS relative entropy can be rewritten as:

$$\hat{D}(\rho||\sigma) = Tr[\sigma(\sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}}) \log(\sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}})]$$

Results for the BS relative entropy

We compute again the entropy production for the BS relative entropy:

Definition The entropy production for BS relative entropy is defined as $EP_{BS} = -\frac{d}{dt}\hat{D}(\rho_t||\sigma)\Big|_{t=0}$:

And we perform the (not so trivial) derivative:

$$EP_{BS} = -Tr[\sigma^{\frac{1}{2}}\mathcal{L}(\rho)\sigma^{-\frac{1}{2}}\log(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}})] - \\ -\int_0^\infty ds Tr[\sigma\rho(\rho+s\sigma)^{-1}\mathcal{L}(\rho)(\rho+s\sigma)^{-1}]$$

Results for the BS relative entropy

With this expression we can finally find the log-Sobolev constant:

Definition The log-Sobolev constant for BS is defined as $\beta(\mathcal{L}) = \inf_\rho \frac{EP_{BS}}{2\hat{D}(\rho||\sigma)}$:

$$\beta(\mathcal{L}) = \inf_\rho \frac{-Tr[\sigma^{\frac{1}{2}}\mathcal{L}(\rho)\sigma^{-\frac{1}{2}}\log(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}})] - \int_0^\infty ds Tr[\sigma\rho(\rho+s\sigma)^{-1}\mathcal{L}(\rho)(\rho+s\sigma)^{-1}]}{2\hat{D}(\rho||\sigma)}$$

This expression is even more cumbersome, so we need to work within particular cases.

Results for the BS relative entropy

For the particular Linbladian $\mathcal{L}(\rho) = \sigma - \rho$ we find an encouraging result:

Theorem For $\mathcal{L} = \sigma - \rho$, we find a lower bound for the BS log-Sobolev constant:

$$\beta(\mathcal{L}) \geq \frac{1}{2}$$

The above Linbladian corresponds to an unipartite system, motivated by the result we may try to find an equivalent expression for bipartite systems.

Results for the BS relative entropy

Some definitions:

Given a bipartite Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ we define:

$$\sigma_{AB} = \sigma_A \otimes \sigma_B; \quad \rho_{AB} \neq \rho_A \otimes \rho_B; \quad \text{and} \quad \rho_{A,B} = Tr_{B,A}[\rho_{AB}]$$

We will also choose a Lindbladian of the form:

$$\mathcal{L}_A = \sigma_A \otimes \rho_B - \rho_{AB}$$

Definition The conditional BS relative entropy takes the form:

$$\hat{D}_A(\rho_{AB}||\sigma_{AB}) = \hat{D}(\rho_{AB}||\sigma_{AB}) - \hat{D}(\rho_B||\sigma_B)$$

Results for the BS relative entropy

We can define the conditional log-Sobolev constant:

If we substitute the Lindbladian in the above equation, we obtain:

$$\beta(\mathcal{L}_A) = \frac{1}{2} + \inf_{\rho_{AB}} \frac{\hat{D}(\rho_B||\sigma_B) - Tr[(\sigma_A \otimes \rho_B)\log(\sigma_{AB}^{-1}\rho_{AB})] - y}{2\hat{D}_A(\rho_{AB}||\sigma_{AB})}$$

Where

$$y = \int_0^\infty ds Tr[\sigma_{AB}\rho_{AB}(\rho_{AB} + s\sigma_{AB})^{-1}(\sigma_A \otimes \rho_B - \rho_{AB})(\rho_{AB} + s\sigma_{AB})^{-1}]$$

It is still not clear if or why this quantity should be greater than 1/2.

Contents

Conclusions and outlook

Conclusions

- We have presented new results for the geometric Rényi divergence and BS relative entropy.
- These quantities, among other divergences, play a key role in characterizing the dynamics of open quantum systems.
- Some lesser used quantities, such as the BS relative entropy, offer a large number of unexplored properties.

Outlook

- Currently, the main quest is to determine whether or not the conditional log-Sobolev constant is greater than 1/2 or even positive for the BS relative entropy in the case of bipartite and multipartite systems with unentangled endpoint σ.
- One can then try to tackle the global log-Sobolev constant.

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Thank you! Questions?