Combining Physics Informed Neural Networks with Learned Regularisation[1]

Inverse problems occupy a central role in scientific imaging, aiming to reconstruct an unknown parameter $x \in X$ from indirect and noisy measurements

$$y = \mathcal{A}x + e \in Y, \tag{1}$$

where X and Y are Hilbert spaces, and $\mathcal{A} : X \to Y$ is a forward operator, in our case the Radon transform. However, equation (1) is frequently *ill-posed*. One solution to this problem is the use of *variational regularisation*

$$\underset{x}{\operatorname{argmin}} ||\mathcal{A}x - y||_2^2 + \lambda f(x), \tag{2}$$

where $f : X \to \mathbb{R}$ is the so-called regulariser and the parameter $\lambda > 0$ balances data fidelity with the regularisation penalty. This regulariser allows the user to incorporate prior information about x. While traditional approaches use *knowledge-driven* regularisers, the advancements of deep learning suggest to learn a regulariser from the data, the *data-driven* regulariser. To accomplish this, the technique of learning *adversarial regularisers* (ARs) is employed.[2] While the AR is often able to achieve high-quality reconstructions, it lacks provable properties, leading to issues when applied in very ill-posed scenarios, requiring various heuristics, like early stopping, for sustained performance. Such heuristics are oftentimes hard to establish in practice. As a consequence, deep input convex neural networks were used to learn an AR which allowed to devise efficient and provable algorithms for reconstruction.[3] However, convexity is on the other side of the spectrum. By virtue of strict guarantees, one gets sustainable performance in the very ill-posed problems, but overall worse numerical performance. Therefore, the convex non-convex network is of particular interest. Here, the regulariser is kept non-convex in a sufficiently structured way to guarantee the convexity of the overall objective. This is accomplished by choosing the regulariser $\mathcal{R}(x) := \mathcal{R}^{cnc}(x, \mathcal{A}x)$ as a combination of a weakly convex function over the data space and a convex function over the parameter space,[4] where

$$\mathcal{R}^{cnc}(x, y) := \mathcal{R}^{c}(x) + \mathcal{R}^{wc}(y). \tag{3}$$

The structural similarity index measure (SSIM) is an index frequently used in medical imaging to evaluate the quality of the reconstruction and takes values in the interval $[0, 1]$, where 1 corresponds to perfect similarity. Figure 1 shows that although all reconstruction methods already perform very well, there is still some room for improvement. Existing methods often treat the optimisation process in isolation from the learned regulariser itself, employing classical descent schemes. This project aimed to address this limitation by incorporating physics-informed neural networks (PINNs) into the regulariser learning. PINNs are leveraged to solve the underlying Hamilton-Jacobi equation alongside learning the regulariser, providing access to the *Moreau envelope* of the regulariser at no extra inference cost. The underlying idea is that the proximal mapping of a function

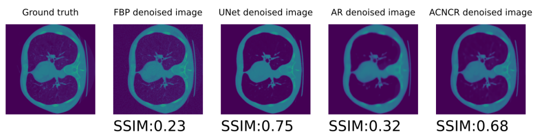

Figure 1: Reconstructed images along with the corresponding SSIM.

$$\operatorname{prox}_{tf}(x) := \underset{z \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ \frac{1}{2t} ||x - z||^2 + f(z) \right\} \tag{4}$$

can be approximated by $\operatorname{prox}_{tf}(x) = x - t\nabla u(x, t)$, where

$$u(t, x) := \min_{z \in \mathbb{R}^n} \left\{ f(z) + \frac{1}{2t} ||x - z||^2 \right\} \tag{5}$$

is the Moreau envelope of the regulariser.[5] This Moreau envelope is a special form of the *Hopf-Lax formula* which gives a solution to the Hamilton-Jacobi equation

$$\begin{cases} u_t + \frac{1}{2}||\nabla u||^2 = 0, & \text{in } \mathbb{R}^n \times (0, T] \\ u = f, & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}. \tag{6}$$

Access to the envelope will not only lead to faster reconstructions, but would also provide ways to find global solutions in non-convex settings. To adapt the ACR network architecture accordingly, we will use insights from low-dimensional PINN examples.

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