Asymptopia

CENTRE FOR MATHEMATICAL SCIENCES NEWSLETTER

UNIVERSITY OF CAMBRIDGE

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Adding prime numbers

By Ben Green

The prime numbers 2, 3, 5, 7, 11, 13, 17, 19, . . . are the building blocks of multiplication. Every number can be written as a product of primes (for example 1001 = 7 x 11 x 13) in precisely one way.

As soon as one tries to combine the primes and the other fundamental operation of arithmetic, namely addition, one swiftly comes up with questions that have proved impossible to answer.

Most famous amongst these is *Goldbach's Conjecture*, which hypothesises that every even number is the sum of two primes. Given a smallish even number it is easy to check that it is the sum of two primes, for example 104 = 31 + 73. To a mathematician, of course, checking that every even number less than 100,000,000 is a sum of two primes is not good enough, and we demand a proof that Goldbach's Conjecture holds for all numbers. Such a proof is currently lacking and most experts do not expect definitive progress on the problem in the near future.

There is a weaker version of Goldbach's Conjecture, which has been proven. Namely, it is known that every sufficiently large *odd* number is the sum of *three* prime numbers. Cambridge mathematicians G. H. Hardy and J. E. Littlewood were the first to plot a path to a proof of this result. It is known that "sufficiently large" in this context can be taken to mean greater than 10^{43001}. This is a number so large that it is of mathematical, rather than merely computational, interest to show that *every* odd number is the sum of three primes.

In April 2004 Terence Tao and I proved a new result in the additive theory of prime numbers (this result was a small part of the reason for Tao receiving the Fields Medal in 2006). Our result concerns arithmetic progressions, or equally spaced sequences, of prime numbers. For example the sequence

199, 409, 619, 829, 1039, 1249, 1459, 1669, 1879, 2089

consists entirely of prime numbers, and the elements of this sequence are equally spaced (the difference between any two adjacent ones is 210). This sequence has length 10. Our result is that one can find such a sequence of any length one desires.

The Green-Tao Theorem: For any natural number k there is an equally spaced sequence of k distinct prime numbers.

Ben Green

The reader may be interested to hear that the longest known explicit arithmetic progression of primes has length 24, and was discovered on 18th January 2007 by Jarosław Wróblewski, using 75 computers at Wrocław University.

At first sight our theorem does not have much to do with adding prime numbers.

However, to say that the primes $p_1, p_2, \ldots, p_n$ lie in arithmetic progression is equivalent to demanding that they satisfy the system of additive equations

$$p_1+p_3 = 2p_2,\ p_2+p_4 = 2p_3,\ \ldots,\ p_{n-2}+p_n = 2p_{n-1}.$$

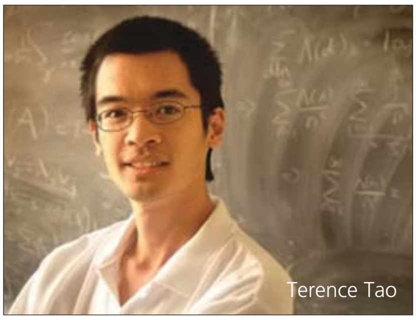
For those keen on linear algebra we note that this is a linear system of equations defined by an $(n-2) \times n$ matrix. By contrast Hardy and Littlewood's equation $p_1 + p_2 + p_3 = N$ is defined by a 1×3 matrix. The height of the matrix is in some sense a measure of the difficulty of the problem, except in the case of the Goldbach Conjecture which, though defined by a mere 1×2 matrix, is degenerate in a certain technical sense and thus presumed to be very hard.

We deduce our theorem from a very famous theorem of the Hungarian mathematician Endre Szemerédi.

Szemerédi's Theorem: Let A be a set of natural numbers with positive upper density. Then, for every k, A contains an arithmetic progression of length k.

The notion of positive upper density is slightly technical, but one should think of it as saying (for example) that *A* contains at least 0.01% of the integers, where the 0.01 could be replaced by 0.00001 or even 0.000000001 if one so desired.

Unfortunately Szemerédi's Theorem does not tell us anything about the set ***P*** of prime numbers, since the primes have upper density zero. This is a well-known fact: the further one looks along the integers, the sparser the primes become. In our work we got around this problem by proving a generalisation of Szemerédi's theorem. Instead of demanding that A have positive upper density, we were able to work with a weaker condition. This is that A has positive upper density relative to some sequence S with a special (and rather technical) property that we term *pseudorandomness*.



Terence Tao

Relative Szemerédi's Theorem: Let A be a set of natural numbers with positive upper density relative to some pseudorandom set S. Then, for every k, A contains an arithmetic progression of length k.

This reduces the task to the construction of a pseudorandom sequence S such that at least 0.0001% (say) of the elements of S are primes. Roughly speaking, one takes S to be the set of *almost primes* – the numbers with at most 100 prime factors, say.

I would like to conclude this article by saying one or two things about where this research has gone in the intervening three years, and what directions we are headed in now. Our next goal is to handle more or less arbitrary systems of linear equations in primes, defined by a general $s \times t$ matrix. The $(n - 2) \times n$ matrix defining an arithmetic progression turned out to be rather special, and to deal with the general case we need to develop completely different techniques.

Hardy and Littlewood's method is well-suited to the case $s = 1$, and the work of another Cambridge mathematician, Timothy Gowers, has contributed to handling systems of equations defined by bigger matrices, say when $s = 2$. But for matrices with 3 or more rows we still only have conjectures rather than theorems. Proving these remains a task for the future.

Ben Green is Herchel Smith Professor of Pure Mathematics at DPMMS.

Global climate change and Arctic sea ice

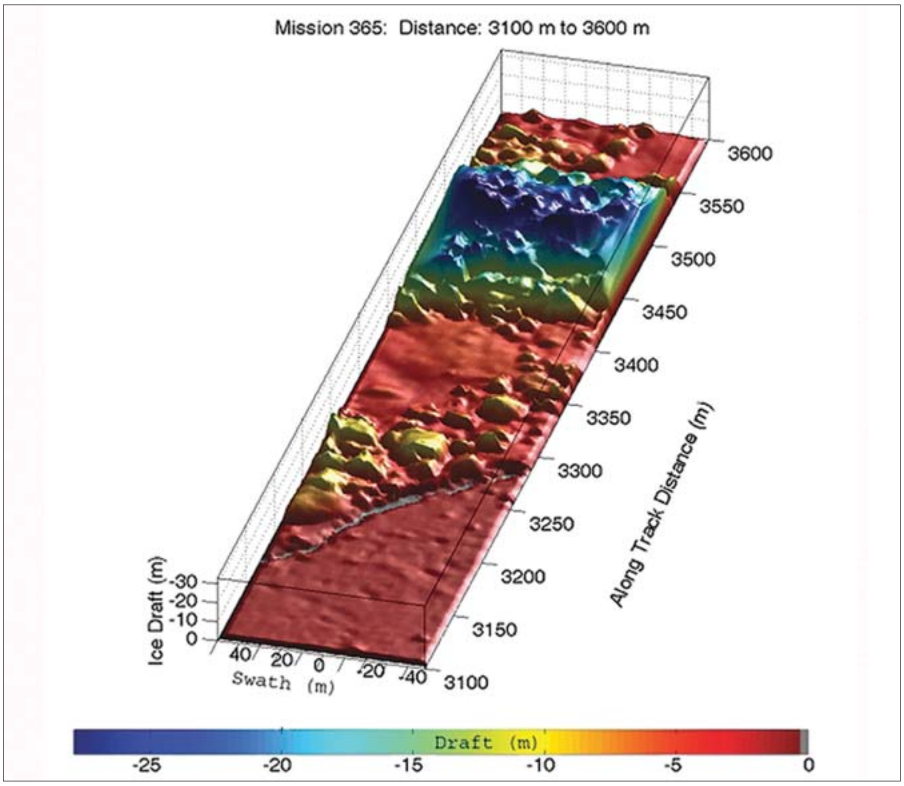
By Peter Wadhams

Evidence for recent warming of the Earth's climate is now very strong. Not only have global-mean temperatures clearly risen over the past century, but the statistics of extreme events, including the intensity and length of droughts and the number of intense hurricanes, show rising trends. Some of the clearest evidence of change is in the Arctic, where average temperatures have increased at twice the global mean rate. Particularly striking is the retreat and thinning of the Arctic sea ice, which has been monitored from satellites.

Arctic sea ice occupies an area of 16 million km^2 in winter, and 8 million in summer. Passive microwave data from satellites has shown us that the area is shrinking by some 3% per decade, accelerating recently to 4%. Last summer had the lowest extent ever, only 5.4 million km^2. The shrinkage rate will cause the summer sea ice cover to disappear by 2080-2100, a conclusion which matches the predictions of climate models. But there is much evidence that the ice will disappear earlier than this. This evidence comes from measurements of ice thickness.

It is very difficult to measure ice thickness synoptically from satellites. The best solution has proved to be measuring from below using submarines. Since 1958 the US have been sending submarines under the Arctic ice and profiling the draft of the ice using upward looking sonar. They were joined in 1971 by the UK, and since that time every UK submarine voyage to the Arctic has included a scientist in its complement, usually myself. This work began at the Scott Polar Research Institute and continued in DAMTP when my sea ice research group moved here in 2002.

Our work involves not only collecting sonar and other data aboard the submarine, but also working with the Navy to plan the submarine's track so that it can collaborate with measurements done from the surface. In different years we have worked with aircraft carrying measuring



A 3D image of the keel of a pressure ridge obtained by upward-looking multibeam echo sounder.

equipment, or with groups working from ice camps. By comparing measurements from the top and bottom of the ice we have been able to gain new insight into the meaning of data from airborne and surface instruments.

The most startling result of the work of our US colleagues and ourselves was our independent discoveries that the Arctic ice has been undergoing dramatic thinning. Comparison of data from the 1970s and 1990s shows a mean loss of more than 40% of thickness during this period. The US and the UK data covered different regions of the Arctic but gave the same percentage loss of thickness.

Why is this happening? There are several competing mechanisms: increased bottom melt, because of a greater influx of warm water into the Arctic from the Atlantic Ocean; reduced winter growth rates, because of warmer air temperatures; a lengthened summer melt period; or a change in ice dynamics because of an altered atmospheric pressure field over the Arctic. One thing that is clear from the data is that pressure ridges, those formidable obstacles to shipping that comprise about half the ice volume in the Arctic, have themselves dramatically decreased in numbers, by 75% in 20 years. As a result, icebreakers can now easily reach the North Pole in summer, when once it was completely inaccessible. The thinning rate is faster than the shrinkage rate, so the Arctic ice cover, instead of being a thick saucer of ice over the polar ocean, is becoming more like a fragile eggshell. It will disappear from the bottom up, either through mechanical failure when the ice is too thin to sustain wind and current stresses, or thermodynamically when the summer melt exceeds the winter growth. Some modellers think that this may happen as early as the 2030s.

Aware of the importance of submarine data for climate analysis, the Royal Navy recently gave us a welcome but unexpected present – a number of sonar datasets from voyages to the Arctic of which we had not been aware, because they involved classified aspects. This raw data must reside in a secure facility – a locked cabinet within a locked office – and initial digitisation must take place within that office. But once the data are digitised, the statistical analysis and interpretation can be done in the real world and freely published. With the aid of support from the Isaac Newton Fund we have begun this process, and will add to the stock by a new voyage to the Arctic this spring.

Another quite different Arctic climate change phenomenon is also a topic for research at DAMTP. This is the role of convective chimneys. In the central Greenland Sea there is a small region where in winter the surface water is made more dense by cooling and by ice formation, and sinks to help drive a global current system known as the thermohaline circulation or the *Great Conveyor Belt*. During several winters, we discovered that the sinking occurs within narrow rotating columns of water, uniform in properties from the surface down to 2500 metres, known as *chimneys*. These were predicted by Peter Killworth of DAMTP in 1979 and their properties deduced. What he could not predict was their long life – we followed one chimney through three years of alternating winter opening and summer closing, an extraordinary longevity for such a small feature. We are finding that as the climate warms and ice fails to form in this region, the driving force for the sinking is diminished. The volume of convection is reduced, and one consequence might be a slowing of the thermohaline circulation – hence the recent headlines about fears of the Gulf Stream slowing down with a cooling effect on European climate. If we can understand chimneys better we can be more certain of what is going to happen in the Greenland Sea and whether such dire consequences will follow.

Peter Wadhams

Peter Wadhams is Professor of Ocean Physics at DAMTP. On one of his recent research trips he was under the ice on the submarine HMS Tireless when an explosion killed two members of the crew. Thankfully, Professor Wadhams and his colleague Nick Hughes returned safely to Cambridge.

People at the CMS
Adrian Potter

Adrian Potter is a Part II mathematician at Pembroke College

I've been studying maths at Cambridge now for a little over two years and I find that people's reaction to this is rather predictable. Amongst Cambridge students the preferred reaction is "oh" and when talking to anyone else it usually is "oh God". The reasons for this are quite simple: Cambridge's impressive reputation and the perceived inaccessibility of maths. However, few people ever experience what it is really like to study maths and to study it here.

Whilst you might get caught up with the many distractions Cambridge has to offer – rowing, drama, nightlife – from the moment you first walk through your college gates you are above all else a maths student. In the beginning the faculty seems somewhat ethereal; their presence being practically unnoticeable beyond lectures and exams. After the first two years of what might be described as "the basics", I have found the third year to be the most stimulating so far. It is now that the maths department suddenly comes out of the shadows and becomes a much more prominent part of your life. For starters you are suddenly at the rather unmistakable CMS every day, but more than that, with so much more freedom to choose what you are studying, everything becomes a much more interactive experience and the course really becomes your own.

As the terms have passed by I have loved every moment in Cambridge. The one pivotal piece of advice I would give to prospective maths students is this: prepare for getting stuck, getting stressed and being wrong. This has certainly been one of the more overwhelming features of my studies. The experience of entering a community where you are so much more average than anywhere you have been before can be quite troubling.

I hope that a year from now I will have more to tell about the new things to be discovered in Part III and beyond. I hope to uncover a range of mysteries, including why one needs a shower next door to the office, how many security doors there are at the CMS and how one institution can employ such a strange variety of lecturers. But for now I'd like to say that the Cambridge Mathematics Tripos deserves all of its reputation and more.

News at the CMS

We congratulate Professor Stephen Hawking FRS on the award of the Royal Society's Copley Medal for his outstanding contribution to theoretical physics and theoretical cosmology. Established in 1731, the Copley Medal is the world's oldest science prize. Previous laureates include Darwin, Einstein and Crick. In honour of Professor Hawking, the medal was carried on the STS 121 Space Shuttle Mission to the International Space Station in July last year.

A number of other prizes and honours have recently been bestowed on members of the CMS. Professor Herbert Huppert FRS from DAMTP has been awarded the 2007 Murchison Medal by the Geological Society of London for his contribution to the geosciences. Professor John Barrow FRS from DAMTP has been awarded honorary doctorates by the University of Durham and the University of Szczecin in Poland. Ben Green, Herchel Smith Professor of Pure Mathematics at DPMMS, has won the 2006 Philip Leverhulme Prize for Mathematics and Statistics. You can read about his work in his article in this issue of *Asymptopia*. Last but not least, Dr Gabriel Paternain from DPMMS has been awarded the 2007 Pilkington Teaching Prize by the University of Cambridge. Congratulations to all!

The CMS extends a warm welcome to Jacob Rasmussen, appointed as Reader in Pure Mathematics, and to Benjamin Schlein, Nathanael Berestycki, Caucher Birkar, Neshan Wickramasekera and Robert Gramacy, all appointed as Lecturers.

To discuss any aspect of making a donation in support of mathematics at Cambridge, please contact Patrick Hawke-Smith (ph250@cam.ac.uk)

Open House at CMS

Professors Geoffrey Grimmett, Head of DPMMS, and Peter Haynes, Head of DAMTP

We are always pleased to welcome new visitors to the Centre for Mathematical Sciences. The undergraduate term finished on a recent Friday, on the following Monday and Tuesday the central common room area was filled with 150 school children taking part in a national mathematical competition – a formidable array of young brain power.

Maths amazes

The following Saturday visitors of many ages were welcomed to a range of lectures and displays as part of the University of Cambridge National Science Week activities.

Coincidentally, on the same day, we welcomed local residents to a ceremony marking the bicentenary of the abolition of the slave trade. The residents have generously donated an oak tree, now planted at the south-west corner of the CMS site. Where is the CMS connection? The adjoining Clarkson and Wilberforce Roads were named in the 1930s (at the centenary of the abolition of slavery in the British Empire) after Thomas Clarkson and William Wilberforce, both members of St John's College and both important figures in the abolition movement.

A date for your diaries: We invite *Asymptopia* readers, alumni and friends to a day of activities on Saturday 22nd September at the CMS, as part of the University Alumni Weekend. (An information sheet and registration form is enclosed.) This will be a chance for a snapshot view of Cambridge mathematics, to sense the great benefits that generous support for the construction of the CMS site has provided and, we hope, the potential for the future. We look forward to welcoming many of you to the Centre for Mathematical Sciences in September.

Young visitors to the CMS