UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 12 June, 2019 9:00 am to 12:00 pm NST0, CST0

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11X****). Answers to these questions must be tied up in* ***separate*** *bundles, marked* ***R, S, T, V, W, X, Y*** *or* ***Z*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct letter* ***R, S, T, V, W, X, Y*** *or* ***Z*** *written in the section box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Single-sided script paper
Rough paper

SPECIAL REQUIREMENTS

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

For $z \in \mathbb{C}$, express

$$f = \frac{z+1}{z^2+1}$$

as the sum of two partial fractions where the denominator of each is a linear function of z. [2]

2

Determine the general term in the power series in x expanded around $x = 0$ for:

(a) $\exp(x^3)$; [1]

(b) $\int_0^x \exp(y^3)\, dy$. [1]

3

Find the general solution of

$$\frac{dy}{dt} + 4\frac{y}{t} = 3 \,. \qquad [2]$$

4

Consider the integral $I_n = \int (\ln x)^n \, dx$ for $x > 0$ and integer $n \geqslant 0$.

(a) Express I_{n+1} in terms of I_n. [1]

(b) Evaluate I_1. [1]

5

Consider the function $f(x, y) = x^2 - x + xy - 3y - y^2 - 1$.

(a) Find the stationary point of $f(x, y)$. [1]

(b) Classify the stationary point. [1]

6

(a) Express $P(A \cup B \cup C)$ in terms of the probabilities of the individual events A, B, C and the intersections between these events. [1]

(b) If $D \subset (A \cup B \cup C)$, determine $P((A \cup B \cup C)|D)$. [1]

7

Determine the value(s) of λ for which there are non-trivial solutions for $\mathbf{x}$ to

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x} = \lambda \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \mathbf{x}\,.$$ [2]

8

Determine the coefficients a_n and b_n for the Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

of $f(x) = \cos^3 x \sin x$ in the interval $-\pi < x \leqslant \pi$. [*Hint:* $\cos^3 x \sin x = (\cos^2 x)(\cos x \sin x)$.]

[2]

9

A solid sphere of radius a has a density distribution $\rho(r) = 1 + r$, where r is the distance from the centre of the sphere. What is the mass of the sphere? [2]

10

(a) Determine the vector area $\mathbf{S}$ (in Cartesian coordinates) of the outside of the shell defined in spherical polar coordinates (r, θ, ϕ) by $r = a$, $0 \leqslant \theta \leqslant \pi/4$, where $\theta = 0$ is in the direction of the z-axis. [1]

(b) What is the projection of $\mathbf{S}$ in the direction given by the Cartesian vector $\mathbf{q}^T = (1, 1, 1)$? [1]

SECTION B

11X

(a) (i) Draw right-handed Cartesian axes and label the following points:

$$\text{the origin } O\,(0,0,0)\,; \quad A\,(1,0,0)\,; \quad B\,(0,1,0)\,; \quad C\,(0,0,1)\,.$$ [2]

(ii) Draw position vectors $\mathbf{e} = (0, \frac{1}{2}, \frac{1}{2})$, $\mathbf{f} = (\frac{1}{2}, 0, \frac{1}{2})$ and $\mathbf{g} = (\frac{1}{2}, \frac{1}{2}, 0)$, leading to the points E, F and G, respectively. Calculate the volume of the parallelepiped defined by $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$. Explain briefly why the result of this calculation is relevant to the question of whether $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ can be used as a basis. [6]

(iii) Write the vector $\overrightarrow{OA}$ in terms of the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. [2]

(b) Calculate $\hat{\mathbf{n}}$, the unit vector normal to the plane containing the points E, F and G. Write down a vector equation for this plane and hence find its Cartesian equation. Calculate the perpendicular distance from the origin to the plane. [6]

(c) The point R with position vector $\mathbf{r}$ lies on the line EF. Show that

$$\mathbf{r} \times (\mathbf{f} - \mathbf{e}) = \mathbf{e} \times \mathbf{f}\,.$$

Find a formula for the shortest distance from the point C to the line EF. Calculate this distance. [The vector product can be written as either $\mathbf{e} \times \mathbf{f}$ or $\mathbf{e} \wedge \mathbf{f}$.] [4]

12R

(a) Find the stationary points and stationary values of the function

$$f(x,y) = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3\,.$$ [6]

(b) Determine the nature of the stationary points. [4]

(c) Sketch the contours of the function in the range $|x| \leqslant 2$, $|y| \leqslant 2$. [6]

(d) Add arrows to your sketch, showing the direction of the gradient vector $\boldsymbol{\nabla} f$, to highlight the behaviour of the function. [4]

13T

Consider the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 2\alpha x + \alpha^2 y + \alpha z \\ \alpha^2 x + \beta z \\ \alpha x + \beta y^2 \end{pmatrix},$$

where α, β are real parameters.

(a) Evaluate $\int \mathbf{F} \cdot \mathbf{dx}$ along the straight line from (0,0,0) to (1,1,1). [4]

(b) Evaluate $\int \mathbf{F} \cdot \mathbf{dx}$ along the path $\mathbf{x}(t) = (t, t^2, t^3)$, with $0 \leqslant t \leqslant 1$. [6]

(c) Determine the value of β for which $\mathbf{F}$ is a conservative field. [3]

(d) For the value of β determined in part (c), find the scalar field $\Phi(\mathbf{x})$ such that $\mathbf{F} = \boldsymbol{\nabla}\Phi$ and $\Phi(\mathbf{x}) = 0$ at the origin. Determine the value(s) of α such that $\Phi(\mathbf{x}) = 1$ at $(1, 1, 1)$. [7]

14W

Let X and Y be continuous independent random variables distributed according to probability density functions $f(x)$ and $g(y)$, respectively.

(a) Give an expression for the probability $P(a \leqslant X \leqslant b)$, where a and b are real parameters and $a < b$. [2]

(b) Consider the new random variable $U = X + c$, with c being a real constant.

(i) Find $P(a \leqslant U \leqslant b)$. [2]

(ii) By considering $P(u \leqslant U \leqslant u + du)$, find $t(u)$, the probability density function for U. [3]

(c) Consider the random variable $Z = X + Y$.

(i) For a given value of Y, find $P(z \leqslant Z \leqslant z + dz | Y = y)$. [2]

(ii) Show that the probability density function of Z, $h(z)$, is given by

$$h(z) = \int_{-\infty}^{\infty} g(y) f(z - y) dy \,. \quad [3]$$

(d) Let $f(x) = \pi^{-1}(1+x^2)^{-1}$ for $-\infty < x < \infty$ and Y be uniformly distributed between $-1/2$ and $1/2$.

(i) Find the probability density function $g(y)$. [1]

(ii) Find the probability density function $h(z)$ for the random variable Z defined in (c). [2]

(iii) Sketch $h(z)$ and find the most probable and mean values of Z. [5]

15Y

(a) Solve the following ordinary differential equations:

(i) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = 0\,,\qquad$ with $\quad y(0) = \pi \quad$ and $\quad y(-\pi/2) = 1\,;$ [3]

(ii) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 2xe^{-x}\,.$ [6]

(b) Consider the pair of ordinary differential equations:

$$\frac{du}{dt} = -3u + v\,,$$

$$\frac{dv}{dt} = -5u + v\,.$$

(i) Rewrite these equations to obtain a second-order differential equation for $u(t)$. [4]

(ii) Determine the general solution for $u(t)$. [3]

(iii) Find the solution for $u(t)$ satisfying $u(0) = 1$ and $v(0) = 1$. [4]

16V

(a) Evaluate

$$\iint_S (z + y^3)\ \mathrm{d}S\,,$$

where S is the total surface made from the vertical cylinder $x^2 + y^2 = a^2$ with $0 \leqslant z \leqslant b$, the flat disc $x^2 + y^2 \leqslant a^2$ in the $z = b$ plane ($b > a > 0$), and the hemispherical indentation $x^2 + y^2 + z^2 = a^2$ with $z \geqslant 0$. [10]

(b) Calculate the flux of the vector field

$$\mathbf{F} = e^{-x}\,\hat{\mathbf{i}} + \frac{1}{x}\left(\frac{1}{(\ln x)^2 + 1}\right)\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$$

through each of the six faces of the axes-aligned unit cube with two vertices at $(0, 0, 0)$ and $(1, 1, 1)$. Hence determine the total flux out of the cube. [10]

17Z

(a) Consider the equation $\mathbf{x}^T\mathbf{M} = \lambda\mathbf{x}^T$, where $\mathbf{x}^T = (x, y, z)$ and

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Determine the values of λ for which there are non-trivial solutions and determine the corresponding solutions. [4]

(b) The elements of the non-singular matrix $\mathbf{A}$ are given by a_{ij} and those of matrix $\mathbf{B}$ by b_{ij}. Both matrices are $n \times n$, where $n > 1$. Write down the following determinants in terms of $|\mathbf{A}|$ and $|\mathbf{B}|$:

(i) $|\mathbf{AB}|$; [2]

(ii) $|\alpha\mathbf{A}^{-1}|$, for constant α; [2]

(iii) $|\mathbf{C}|$, where the elements of $\mathbf{C}$ are related to those of $\mathbf{A}$ through $c_{ij} = \beta_i\gamma_j a_{ij}$ and β_i, γ_j are a set of constants. [3]

(c) The $n \times n$ matrix $\mathbf{Q}$ (with $n > 1$) has eigenvalues λ_i with corresponding eigenvectors $\mathbf{e}_i$. From these we define two additional matrices, $\mathbf{D}$ and $\mathbf{E}$. The matrix $\mathbf{D}$ is diagonal with non-zero elements $d_{ii} = \lambda_i$. The columns of the matrix $\mathbf{E}$ are the eigenvectors (i.e. $\mathbf{E} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$). You may assume $|\mathbf{E}| \neq 0$.

(i) Express $\mathbf{Q}$ in terms of $\mathbf{D}$ and $\mathbf{E}$. [4]

(ii) The $m \times n$ matrix $\mathbf{V}_k$ is determined recursively by $\mathbf{V}_k = \mathbf{V}_{k-1}\mathbf{Q}$, for $k \geqslant 1$. Express $\mathbf{V}_k$ in terms of $\mathbf{V}_0$, $\mathbf{D}$ and $\mathbf{E}$. [5]

18S

Consider the periodic functions $f(x)$ and $g(x)$, both with period 2, defined over the range $-1 < x \leqslant 1$ as $f(x) = \cosh x$ and $g(x) = \sinh x$.

(a) Determine the Fourier Series representation of $f(x)$. Hence or otherwise, show that $f(1)$ can be written in the form

$$\cosh(1) = \sinh(1)\left(1 + p\sum_{n=1}^{\infty}\frac{1}{1+n^2q^2}\right),$$

and determine the constants p and q. [8]

(b) Determine the Fourier Series representation of $g(x)$. [*Hint: This can be achieved without computing any further integrals.*] [4]

(c) State Parseval's theorem and use it to show that $\int_{-1}^{1}(f(x) - g(x))^2\,dx = \sinh(2)$. [8]

19T*

(a) Use Lagrange multipliers to determine the height and radius of the circular cylinder (with ends normal to the axis of the cylinder) of volume V that has the minimal total surface area S and calculate this area. [6]

(b) Consider a cone of height h and base radius r. The axis of the cone is aligned with the z-axis and the base is normal to the z-axis. The apex of the cone, located at $(0, 0, R)$ with constant R, lies on the surface of a sphere that is centred on the origin O. Using a Lagrange multiplier λ to enforce the condition that the cone is inscribed by the sphere (see figure), determine the values of r and h that maximise the volume of the cone. Calculate the maximum volume of the cone as a fraction of the volume of the sphere. [*Hint: the volume of the cone is given by* $V = \frac{\pi}{3}r^2h$.] [6]

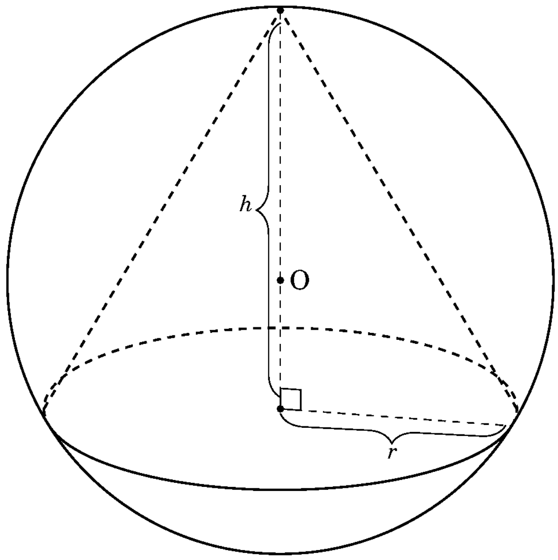

Figure: Sketch of a cone of height h and base radius r inscribed into a sphere centred on O.

(c) Find the minimum of the function

$$f(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_i, \qquad \text{with} \quad x_i > 0 \quad \text{and} \quad i = 1, ..., n,$$

subject to constraint

$$\prod_{i=1}^{n} x_i = a, \qquad \text{with constant } a \in \mathbb{R}\,,\ a > 0.$$

Hence or otherwise, deduce an inequality relating the arithmetic mean $\frac{1}{n}\sum_{i=1}^{n} x_i$ and geometric mean $\left(\prod_{i=1}^{n} x_i\right)^{1/n}$. [8]

20Y*

(a) By using the method of separation of variables, find a non-trivial solution for each of the following first-order partial differential equations for real-valued functions $u(x, y)$:

(i) $\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 0\,;$ [3]

(ii) $x\dfrac{\partial u}{\partial x} - 2y\dfrac{\partial u}{\partial y} = 0\,.$ [3]

(b) A function $T(x, t)$ obeys the diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \qquad (\dagger)$$

for a constant $\kappa > 0$, non-negative t and $-\infty < x < \infty$. The initial condition is given by $T(x, t = 0) = T_0 \exp\left(-x^2/L^2\right)$, where T_0 and L are positive constants.

Substitute a solution of the form $T(x, t) = F(t) \exp(-x^2 H(t))$ into equation (†), where $F(t)$ and $H(t)$ are positive differentiable functions, to obtain two simultaneous equations containing $F(t)$ and $H(t)$. Solve these equations for $F(t)$ and $H(t)$. [4]

Thus show that

$$H(t) = 1/(4\kappa t + L^2) \qquad \text{and} \qquad F(t) = T_0 L/\sqrt{4\kappa t + L^2}\,. \qquad [8]$$

Write out the solution of equation (†). [2]

END OF PAPER