UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Tuesday, 29 May, 2018 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***3C****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
3 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

(a) Show that, for vector fields $\mathbf{a}$ and $\mathbf{b}$ in three dimensions,

$$\mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{b}) = \mathbf{b} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) - \boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{b}) \ .$$

[4]

(b) By applying the divergence theorem to a vector field $\mathbf{a} \times \mathbf{F}$, where $\mathbf{a}$ is an arbitrary constant vector, show that

$$\int_V \boldsymbol{\nabla} \times \mathbf{F} \, dV = - \int_S \mathbf{F} \times \mathbf{d}\boldsymbol{S} \, , \qquad (\star)$$

when S is the surface of a closed volume V. [4]

(c) Suppose now that V is the hemisphere $\{|\boldsymbol{x}| \leqslant R, \ z \geqslant 0\}$ with $R > 0$, and $\mathbf{F} = (z, x^2 + y^2, 0)$ in Cartesian coordinates. Verify the equality in $(\star)$ by calculating both integrals and showing that they take the form

$$\begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix}$$

in Cartesian coordinates, for some constant α that you should determine. [12]

2C

Let $u(x,t)$ denote the displacement of a string that is stretched horizontally between $x = 0$ and $x = L > 0$, and fixed at these points such that $u(0,t) = u(L,t) = 0$. The displacement of the string is subject to a resistance that is proportional to its velocity. In terms of scaled variables, the displacement satisfies

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - 2\lambda \frac{\partial u}{\partial t},$$

where

$$\lambda = \frac{\pi}{L}\left(M + \frac{1}{2}\right)$$

is the resistance coefficient, and $M > 0$ is an integer.

(a) By using separation of variables, write down ordinary differential equations for the spatial and temporal dependence of the displacement, respectively. [4]

(b) The string is initially horizontal, and is subject to an impulsive initial velocity $\partial u/\partial t = f(x)$ at $t = 0$. Show that the general solution for the displacement can be written in the following form:

$$u(x,t) = e^{-\lambda t}\left[\sum_{n=1}^{M} A_n \sin(\alpha_n x)\sinh(\Omega_n t) + \sum_{n=M+1}^{\infty} A_n \sin(\alpha_n x)\sin(\omega_n t)\right].$$

Give expressions for each of α_n, Ω_n and ω_n, in terms of M and L. [8]

(c) Suppose now that $L = \pi$ and $M = 3$. Calculate the coefficients A_n when $f(x) = e^{-x}$.

[Hint: You may find it helpful to use integration by parts to generate a recurrence relation.] [8]

3C

(a) Calculate the Green's function $G(x, \xi)$ that satisfies

$$\frac{\mathrm{d}^2 G}{\mathrm{d}x^2} - G = \delta(x - \xi),$$

on the interval $0 \leqslant x < \infty$, subject to the boundary condition $G(0, \xi) = 0$, with G remaining bounded as $x \to \infty$. Hence write down, in integral form, the bounded solution $y(x)$ of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y = f(x), \qquad 0 \leqslant x < \infty, \qquad y(0) = 0. \qquad (\dagger)$$

[8]

(b) Consider now the problem

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}u}{\mathrm{d}x} - u = g(x), \qquad 0 \leqslant x < \infty, \qquad (*)$$

where $u(x)$ remains bounded throughout the domain.

(i) Show, by means of the substitution $u = y/x$, that $(*)$ reduces to the form of $(\dagger)$. [2]

(ii) Show further, using the Green's function calculated in part (a), or otherwise, that the solution of $(*)$ in the case

$$g(x) = \begin{cases} 0 & 0 \leqslant x < 1, \\ 1 & x \geqslant 1, \end{cases}$$

is

$$u(x) = \begin{cases} a(x) & 0 \leqslant x < 1, \\ -1 + A\, e^{-x}/x & x \geqslant 1, \end{cases}$$

where the function $a(x)$ and constant A should be specified. [10]

4A

The Fourier transform of a function $f(t)$ is given by

$$\widetilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt.$$

(a) What is the Fourier transform of $\widetilde{f}(t)$ in terms of f (the duality property)? [2]

(b) What is the Fourier transform of $f(bt)$ in terms of $\widetilde{f}$, where b is a constant (the scaling property)? [2]

(c) Derive the Fourier transform of $g(t) = e^{-\alpha t} u(t)$, where $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t \geqslant 0$, and α is a constant. [3]

(d) Using the linearity property of the Fourier transform, together with parts (b) and (c), determine the Fourier transform of $h(t) = e^{-\alpha|t|}$. [3]

(e) Use parts (a) and (d) to determine the Fourier transform of $s(t) = (1 + t^2)^{-1}$. [3]

(f) Find the Fourier transform of $v(t, T) = \frac{1}{2}(u(t + T) - u(t - T))$, where u is defined in part (c) and T is a constant. [3]

(g) A signal $z(t)$ is given by

$$z(t) = \frac{\sin t}{\pi t} + \frac{\sin 2t}{\pi t}.$$

Plot the graph of $|\widetilde{z}(\omega)|^2$ versus ω and use Parseval's theorem to find the energy E of the signal $z(t)$ defined as

$$E = \int_{-\infty}^{\infty} |z(t)|^2\, dt.$$

[4]

5B

Given a square matrix $\mathbf{A}$ we define the matrix exponential via the formula:

$$\exp(\mathbf{A}) = \mathbf{I} + \sum_{k=1}^{\infty} \frac{1}{k!}\mathbf{A}^k.$$

where $\mathbf{I}$ is the identity matrix.

(a) Show that for any invertible matrix $\mathbf{P}$, $\exp(\mathbf{PAP}^{-1}) = \mathbf{P}\exp(\mathbf{A})\mathbf{P}^{-1}$. [6]

(b) Show that if $\mathbf{A}$ is skew-Hermitian, then $\exp(\mathbf{A}) = \mathbf{QCQ}^\dagger$ where $\mathbf{Q}$ is a unitary matrix, and $\mathbf{C}$ is a diagonal matrix with complex numbers of unit modulus on the diagonal. Deduce that $\exp(\mathbf{A})$ is unitary. [7]

(c) Show that if $\mathbf{A}$ is Hermitian, then $\exp(\mathbf{A})$ is a Hermitian matrix with positive eigenvalues. [7]

6B

(a) State the definition of a *diagonalizable* matrix. Give an example of a 2×2 diagonalizable matrix, and an example of a 2×2 non-diagonalizable matrix. [5]

(b) Find the eigenvalues of the following symmetric matrix: [6]

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

[*Hint: Consider the matrix vector product* $\mathbf{M}\left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]$.]

(c) Show that the set of $\mathbf{x} = (x_1, x_2, x_3)$ that satisfy $\mathbf{x}^T\mathbf{M}\mathbf{x} = 0$ and $x_1 + x_2 + x_3 = 0$ consists of two infinite lines, where $\mathbf{M}$ is the matrix from part (b) (you do not need to find the equations of these lines).

[*Hint: Write* $\mathbf{M} = \mathbf{PDP}^T$ *with* $\mathbf{D}$ *diagonal, and* $\mathbf{P} = [\mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \mathbf{u}_3]$ *where* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *are the eigenvectors of* $\mathbf{M}$.] [9]

7A

(a) Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function of $z = x + iy$ for real x, y, u, v. Prove that u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Define a *harmonic function*.

Verify that $u(x, y) = \ln(x^2 + y^2)$ defined on the complex plane with the origin removed is harmonic and find a conjugate harmonic function $v(x, y)$ (i.e. v such that $u + iv$ is analytic). [8]

(b) Find a power-series expansion of the function $f(z) = (3 - z)^{-1}$ about the point $z = 4i$, and calculate the radius of convergence. [5]

(c) Find a power-series expansion of the function $g(z) = (1 - z^2)\exp(1/z)$ about $z = 0$. Determine whether $z = 0$ is a pole or an essential singularity. Compute the residue at $z = 0$. [7]

8B

Consider the second-order differential equation:

$$2x^2 y'' - xy' + (1 + x)y = 0.$$

(a) Show that $x = 0$ is a regular singular point. [2]

(b) Consider a solution of the form

$$y(x) = x^\sigma \sum_{n=0}^{\infty} a_n x^n, \quad (a_0 \neq 0).$$

Determine the two possible values of σ for such a solution to exist. [6]

(c) For each value of σ determine the recursion relations satisfied by the a_n. Solve the recursion relations and express a_n in terms of a_0 in each case. [8]

(d) Find the radius of convergence of the power series solutions in each case. [4]

9C

(a) State the Euler–Lagrange equation for the extrema of the functional

$$T[y] = \int_a^b f(x, y, y')\, dx\,.$$

Write down the integral of the Euler–Lagrange equation if $f = f(x, y')$ does not depend explicitly on $y(x)$. [3]

(b) Light travels in a plane with refractive index $\mu(x)$, given in a piecewise manner by

$$\mu^2 = \begin{cases} 1 + (1-x)^{2/3} & x < 0 \\ 2 + x & x \geqslant 0. \end{cases}$$

A light ray is fired from a point $(x, y) = (-\alpha, 0)$, where α is a positive constant, and is picked up by a receiver on the line $x = \alpha$. The travel time of the ray along a path $y(x)$ is given by the functional

$$T[y] = \int_{-\alpha}^{\alpha} \mu(x)\sqrt{1 + y'^2}\, \mathrm{d}x\,.$$

Suppose that the ray crosses $x = 0$ with a slope $y' = 1$.

(i) Determine a piecewise expression for the slope $y'(x)$ of the path of least time for the ray. [4]

(ii) Hence calculate the path of least time $y(x)$ in terms of α. [4]

(iii) Suppose that the slope of the light ray at the receiver $x = \alpha$ is half of its initial slope. Determine the value of α in this case. [2]

(iv) Suppose instead that the light ray is fired with an initial slope $y' = 1/2$. Determine the value of α in this case, and find and sketch the path of the light ray. At what value of y does the ray hit the receiver? [7]

10C

A plate with thermal diffusivity $\kappa(r)$ occupies the region $r \leqslant 1$, where r is the radial polar co-ordinate. The temperature $T(r,t)$ of the plate is axisymmetric and satisfies

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\kappa(r)\frac{\partial T}{\partial r}\right),$$

with $T = 0$ at $r = 1$.

(a) By looking for separable solutions of the form $T = e^{-\nu t}R(r)$, find an ordinary differential equation satisfied by $R(r)$. Write this equation in Sturm–Liouville form and identify the weight function. [3]

(b) Assuming that $R(r)$ and $\kappa(r)$ remain finite for $r \leqslant 1$, show that

$$\nu \int_0^1 rR^2 \, \mathrm{d}r = \int_0^1 r\kappa(r)R'^2 \, \mathrm{d}r \, . \qquad (\star)$$

[4]

(c) Explain briefly why $(\star)$ can be used to generate an upper bound for the decay rate ν of the fundamental mode. [You may quote the Euler–Lagrange equation without proof.] [4]

(d) Suppose now that $\kappa(r) = r^n$, with $n \geqslant 0$.

(i) Using the trial function $R_{\text{trial}}(r) = 1 - r^q$ with $q > 0$, find an upper bound ν_{trial} for the decay rate of the fundamental mode as a function of q and n. [4]

(ii) If $n = 0$, determine the value of q that yields the best possible bound ν_{trial}, and give the value of that bound. Given that the corresponding trial temperature profile is $T_{\text{trial}}(r,t) = e^{-\nu_{\text{trial}}t}R_{\text{trial}}(r)$, sketch $T_{\text{trial}}(r,0)$ and $T_{\text{trial}}(0,t)$. [5]

END OF PAPER