UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday, 2 June, 2017 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6B****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B

Let $y_n(x)$ and λ_n, $n = 0, 1, 2...$, be the real normalised eigenfunctions and corresponding eigenvalues for the Sturm–Liouville eigenvalue problem

$$\mathcal{L}\, y_n(x) \equiv -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y_n(x)}{\mathrm{d}x}\right] + q(x)y_n(x) = \lambda_n w(x) y_n(x)\,, \quad 0 \leqslant x \leqslant 1, \quad y_n(0) = y_n(1) = 0\,,$$

with $p(x) > 0$ and $w(x) > 0$.

(a) State, without proof, the orthonormality property for two eigenfunctions $y_n(x)$ and $y_m(x)$. [2]

(b) Given the completeness of the eigenfunctions, any real function $f(x)$ satisfying the same boundary conditions as $y_n(x)$ can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n\, y_n(x)\,,$$

for some real constants a_n. Show that

$$\int_0^1 w(x) f(x)^2\, \mathrm{d}x = \sum_{n=0}^{\infty} a_n^2.$$

[5]

(c) Now consider the equation $\mathcal{L}Y(x) = w(x)[\alpha\, Y(x) + f(x)]$, where α is a constant, $\alpha \neq \lambda_n$ for any n, and $Y(x)$ satisfies the same boundary conditions as $y_n(x)$.

(i) Show that the solution of this equation can be written as

$$Y(x) = \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n - \alpha}\, y_n(x)\,.$$

[6]

(ii) Suppose that $\lambda_0 = 1$, $\lambda_1 = 2$, and $f = 2\,[y_0(x) + y_1(x)] - \alpha\, y_1(x)$. Given that

$$\int_0^1 w(x) Y(x)^2\, \mathrm{d}x = 2\,,$$

and that $\alpha > 0$, find α and express $Y(x)$ in terms of the eigenfunctions $y_n(x)$. [7]

2C

Consider the Laplace equation in elliptic coordinates

$$\frac{\partial^2\Phi}{\partial\mu^2}+\frac{\partial^2\Phi}{\partial\nu^2}=0\,, \tag{$\star$}$$

where $\mu > 0$, $0 \leqslant \nu < 2\pi$ is a periodic coordinate and Φ is single valued, so $\Phi(\mu,\nu)=\Phi(\mu,\nu+2\pi)$.

(a) Use separation of variables to show that the general solution of ($\star$) that is continuous and single valued for $\mu > 0$ can be written as

$$\Phi = A_0 + B_0\mu + \sum_{n=1}^{+\infty}\left\{\Big[A_n\cosh(n\mu)+B_n\sinh(n\mu)\Big]\cos(n\nu)\right.$$
$$\left. + \Big[C_n\cosh(n\mu)+D_n\sinh(n\mu)\Big]\sin(n\nu)\right\},$$

where A_n, B_n, C_n and D_n are constants. [10]

(b) A line of constant μ is an ellipse with semi-major axis $\cosh\mu$ and semi-minor axis $\sinh\mu$. Such an ellipse can be defined in terms of Cartesian coordinates as

$$\frac{x^2}{\cosh^2\mu}+\frac{y^2}{\sinh^2\mu}=1\,.$$

The function Φ satisfies ($\star$) in the region defined by $a<\mu<b$. At the inner ellipse, defined by $\mu=a$, Φ has normal derivative

$$\left.\frac{\partial\Phi}{\partial\mu}\right|_{\mu=a}=-\cos(2\nu)\,.$$

The outer ellipse, defined by $\mu=b$, is held at $\Phi(b,\nu)=\cos(\nu)$. Use separation of variables to find Φ in the region $a<\mu<b$. [10]

3C

(a) Let ϕ be a scalar field that tends to zero as $|\mathbf{r}| \to +\infty$ and satisfies the Klein-Gordon equation

$$(\nabla^2 - k^2)\phi = \rho\,,$$

where $\rho(\mathbf{r})$ tends to zero rapidly as $|\mathbf{r}| \to +\infty$ and k is a real constant.

(i) Verify that

$$\phi(\mathbf{r}) = \int_{\mathbb{R}^3} G(\mathbf{r}, \tilde{\mathbf{r}})\rho(\tilde{\mathbf{r}}) \mathrm{d}^3\tilde{\mathbf{r}}\,,$$

where $G(\mathbf{r}, \tilde{\mathbf{r}})$ satisfies

$$(\nabla^2_{\mathbf{r}} - k^2)G(\mathbf{r}, \tilde{\mathbf{r}}) = \delta^{(3)}(\mathbf{r} - \tilde{\mathbf{r}})\,. \qquad (\star)$$

[4]

(ii) Show that

$$G(\mathbf{r}, \tilde{\mathbf{r}}) = A\frac{e^{-k|\mathbf{r}-\tilde{\mathbf{r}}|}}{|\mathbf{r} - \tilde{\mathbf{r}}|}\,,$$

and determine A. [6]

(iii) Let V be the half plane of $\mathbb{R}^3$ with $z > 0$. Use the method of images to determine $G(\mathbf{r}, \tilde{\mathbf{r}})$ satisfying $(\star)$ everywhere on V, subject to $G(z = 0) = 0$, and $G \to 0$ as $|\mathbf{r}| \to +\infty$. [2]

(b) Let V be a region of three-dimensional space with boundary S. The scalar function $\psi(\mathbf{r})$ satisfies Laplace's equation in V

$$\nabla^2\psi = 0\,,$$

and $\psi(\mathbf{r}) = w(\mathbf{r})$ on S, where $w(\mathbf{r})$ is an arbitrary scalar function defined throughout V.

Show that

$$\int_V (\boldsymbol{\nabla} w) \cdot (\boldsymbol{\nabla} w)\, \mathrm{d}^3\mathbf{r} \geqslant \int_V (\boldsymbol{\nabla} \psi) \cdot (\boldsymbol{\nabla} \psi)\, \mathrm{d}^3\mathbf{r}\,.$$

[*Hint: Consider the inequality* $\int_V \boldsymbol{\nabla}(\psi - w) \cdot \boldsymbol{\nabla}(\psi - w)\, \mathrm{d}^3\mathbf{r} \geqslant 0$.] [8]

4A

(a) Prove that if $f(z)$ is analytic and has a simple zero at $z = z_0$ then $1/f(z)$ has a simple pole with residue $1/f'(z_0)$ there. What is the residue of $g(z)/f(z)$ at $z = z_0$ if $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$? [6]

(b) Consider the function

$$h(z) = \frac{1}{a - \frac{1}{2i}(z - z^{-1})},$$

where $a > 1$ and real. State the location of any singularities of $h(z)$ and calculate the residue of $h(z)$ for the singularity that lies inside the unit circle. [6]

(c) Use the result of part (b) and contour integration to evaluate

$$\int_{-\pi}^{\pi} \frac{\sin\theta}{a - \sin\theta}\, d\theta,$$

where $a > 1$ and real. [8]

5A

The Fourier transform of a function $g(t)$ is given by

$$\widetilde{g}(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t}\, \mathrm{d}t.$$

(a) Given the Fourier transform $\widetilde{f}(\omega)$ of the function $f(t)$ derive Fourier transforms of $f'(t)$, $f''(t)$ and $tf(t)$, assuming that $f(t)$, $f'(t) \to 0$ as $|t| \to +\infty$. [6]

(b) Show that the Fourier transform of $f(t) = \exp(-t^2/2)$ satisfies

$$\frac{\mathrm{d}\widetilde{f}}{\mathrm{d}\omega} = h(\omega)\widetilde{f}\,,$$

for some $h(\omega)$ which you should find explicitly. Solve this equation to determine $\widetilde{f}$ up to a multiplicative constant. [6]

(c) Let $y(t)$ satisfy Bessel's equation of order zero, so that

$$t\, y''(t) + y'(t) + t\, y(t) = 0\,.$$

Show that the Fourier transform of $y(t)$ satisfies a first-order differential equation. Solve this equation up to an arbitrary multiplicative constant. Using the inverse Fourier transform, express the Bessel function $y(t)$ in terms of an integral. [8]

6B

(a) State the transformation law for a tensor of order n. Given vectors $\boldsymbol{u}$ and $\boldsymbol{\Omega}$, show that the quantity

$$\boldsymbol{W} = \boldsymbol{u} \times (\boldsymbol{\Omega} \times \boldsymbol{u})$$

also transforms as a vector. [5]

(b) Let V denote the volume inside a sphere of radius a. Explain briefly why the integral

$$\int_V x_{i_1} x_{i_2} ... x_{i_n} \, \mathrm{d}V$$

is an isotropic tensor for any positive integer n. [2]

Hence show that

$$\int_V x_i \, \mathrm{d}V = 0\,, \quad \int_V x_i x_j \, \mathrm{d}V = \alpha \delta_{ij} \quad \text{and} \quad \int_V x_i x_j x_k \, \mathrm{d}V = 0\,,$$

for some constant α to be determined. You may state without proof the form of the general isotropic tensors of order 1, 2 and 3. [5]

(c) Suppose now that $\boldsymbol{\Omega}$ is a constant vector and $u_i = \Omega_i + \beta\, x_i$ for a constant scalar β. Determine $\boldsymbol{W}(\boldsymbol{x})$ and calculate

$$\int_V W_i \, \mathrm{d}V\,.$$

[8]

7A Consider a system consisting of three particles of masses $m_1 = m, m_2 = \mu m$ and $m_3 = m$, connected in that order in a straight line by two equal light springs of force constant k.

(a) Write down the kinetic and potential energies of the system in terms of the coordinates of the particles $x_1(t), x_2(t), x_3(t)$. Write down the corresponding symmetric matrices for the kinetic and potential energies. [6]

Find the normal frequencies of the system and the corresponding normalised eigenvectors. Describe the physical motions associated with these normal modes. [7]

(b) Consider the particular case in which $\mu = 2$. Show that the three normal (angular) frequencies are $0, \Omega$, and $\sqrt{2}\Omega$ where you should specify Ω in terms of k and m. Show that the corresponding (unnormalised) eigenvectors are

$$\mathbf{x}^1 = (1,1,1)^T, \quad \mathbf{x}^2 = (1,0,-1)^T, \quad \mathbf{x}^3 = (1,-1,1)^T.$$

Write down the orthogonality property of these eigenvectors with respect to the kinetic matrix. [3]

(c) The masses are released from rest with initial displacements relative to their equilibrium positions of $x_1 = 2\epsilon$, $x_2 = -\epsilon$ and $x_3 = 0$, for some real constant ϵ. Determine their subsequent motions. [4]

8C

(a) Determine the elements of the cyclic group generated by the matrix

$$\mathsf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

explicitly. [6]

(b) Construct the multiplication table of the following set of matrices, and verify that they form a group under matrix multiplication:

$$\mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathsf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

[6]

(c) Prove that the set of elements of finite order in an Abelian group is a subgroup. [8]

9C

Consider the set of matrices of the form

$$\mathsf{A} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b, c are integers modulo 5 (for example $7 \,\text{modulo}\, 5 = 2$).

(a) Show they form a finite group G under matrix multiplication. Show that G has 125 elements. [7]

(b) Show that the subset given by $a = c$ defines an Abelian subgroup H. Find the order of H and verify Lagrange's theorem. How many distinct left cosets of H are in G? [7]

(c) Find the set of all the elements of G whose square is the 3×3 identity matrix. Is the subset of G defined by $b \neq 0$ a subgroup of G? [6]

10A

(a) Define a *representation* $D = \{D(X)\}$ of a group $\mathcal{G}$ and use the definition to prove that the matrix associated with the inverse of X is the inverse of the matrix associated with X. [4]

(b) A group $\mathcal{G}$ has four elements I, X, Y and Z, which satisfy $X^2 = Y^2 = Z^2 = XYZ = I$. Show that all elements commute with other elements. Deduce the form of the character table of the group $\mathcal{G}$. [9]

(c) For which real numbers p do the matrices

$$D(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(X) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D(Y) = \begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix}, \quad D(Z) = \begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix},$$

form a representation D of $\mathcal{G}$? Find its characters. [7]

END OF PAPER