UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS **Part IB & II (General)**

Friday, 27 May, 2016 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6C****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
3 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B

Consider the Sturm-Liouville system

$$\mathcal{L}y(x) - \lambda\omega(x)y(x) = 0, \quad a \leqslant x \leqslant b,$$

where

$$\mathcal{L}y(x) \equiv -[p(x)y'(x)]' + q(x)y(x)$$

with $\omega(x) > 0$ and $p(x) > 0$ for all x in $[a, b]$. The boundary conditions on y are

$$\begin{aligned} A_1\, y(a) + A_2\, y'(a) &= 0, \\ B_1\, y(b) + B_2\, y'(b) &= 0, \end{aligned}$$

where A_1, A_2, B_1 and B_2 are constants and all functions are real.

(a) Show that with these boundary conditions, $\mathcal{L}$ is self-adjoint. [4]

(b) By considering $y\mathcal{L}y$, or otherwise, show that the eigenvalue λ can be written as

$$\lambda = \frac{\int_a^b [py'^2 + qy^2]dx - [pyy']_a^b}{\int_a^b \omega y^2 dx}.$$

[4]

(c) Now suppose that $a = 0$ and $b = \ell$, that $p(x) = 1$, $q(x) \geqslant 0$ and $\omega(x) = 1$ for all x in $[0, \ell]$, and that $A_1 = 1$, $A_2 = 0$, $B_1 = k > 0$ and $B_2 = 1$. Show that the eigenvalues of this Sturm-Liouville system are strictly positive. [4]

(d) Assume further that $q(x) = 0$ and solve the system explicitly. With the aid of a sketch, show that there exist infinitely many eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$. [6]

(e) Describe the behaviour of λ_n as $n \to \infty$. [2]

2C

Consider the Laplace equation in bipolar coordinates

$$\frac{\partial^2 \Phi}{\partial \sigma^2} + \frac{\partial^2 \Phi}{\partial \tau^2} = 0\,, \qquad (\star)$$

where $0 \leqslant \sigma < 2\pi$ is a periodic coordinate and $\tau > 0$.

(a) Use separation of variables to show that the general solution of $(\star)$, which is continuous and single valued for $\tau > 0$, can be written as

$$\Phi = A_0 + B_0\tau + \sum_{n=1}^{+\infty} \Big\{ \Big[A_n \cosh(n\tau) + B_n \sinh(n\tau) \Big] \cos(n\sigma) + \Big[C_n \cosh(n\tau) + D_n \sinh(n\tau) \Big] \sin(n\sigma) \Big\}\,,$$

where A_n, B_n, C_n and D_n are constants. [10]

(b) A line of constant τ is a circle of radius $1/\sinh\tau$ that can be defined in terms of Cartesian coordinates as

$$y^2 + (x - \coth\tau)^2 = \frac{1}{\sinh^2\tau}\,.$$

Suppose Φ satisfies $(\star)$ in the region defined by $a < \tau < b$. The inner circle, defined by $\tau = a$, is held at $\Phi = 0$ and the outer circle, defined by $\tau = b$, is held at $\Phi = \cos(2\sigma)$. Use separation of variables to find Φ in the region $a < \tau < b$. [10]

3C

Let V be a region of three-dimensional space with boundary S.

(a) Prove that

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \int_S (\phi\,\mathbf{n}\cdot\boldsymbol{\nabla}\psi - \psi\,\mathbf{n}\cdot\boldsymbol{\nabla}\phi)dS\,,$$

where ϕ and ψ are scalar fields and $\mathbf{n}$ is the outwards directed unit normal to S. [3]

(b) Let ϕ be a scalar field that tends to zero as $|\mathbf{r}| \to +\infty$ and satisfies the Poisson equation

$$\nabla^2\phi = -\rho\,,$$

where $\rho(\mathbf{r})$ tends to zero rapidly as $|\mathbf{r}| \to +\infty$.

(i) Show that

$$\phi(\mathbf{r}) = \int_{\mathbb{R}^3} G(\mathbf{r},\tilde{\mathbf{r}})\rho(\tilde{\mathbf{r}})d\tilde{V}\,,$$

where $G(\mathbf{r},\tilde{\mathbf{r}})$ satisfies

$$\nabla^2_{\mathbf{r}}G(\mathbf{r},\tilde{\mathbf{r}}) = -\delta^{(3)}(\mathbf{r}-\tilde{\mathbf{r}})\,.$$

[3]

(ii) Determine $G(\mathbf{r},\tilde{\mathbf{r}})$. [4]

(iii) Show that

$$\phi(\mathbf{r}) = \frac{e^{-\beta|\mathbf{r}|}-1}{|\mathbf{r}|\beta^2}$$

for the case

$$\rho(\mathbf{r}) = -\frac{e^{-\beta|\mathbf{r}|}}{|\mathbf{r}|}\,,$$

where $\beta > 0$. [10]

4A

(a) Consider the integral

$$I = \int_C \frac{f(z)}{\sqrt{z}} dz$$

along some contour C, where $z = re^{i\theta}$ is complex, $f(z)$ is analytic and nonzero along the real axis, and the branch cut associated with the integrand is taken along the positive real axis.

(i) Determine the integral I for the contour C given by $r = R$, followed in a clockwise direction from $\theta = \frac{3}{2}\pi$ to $\theta = \frac{1}{2}\pi$, in the limit $R \to 0$. If $f(z)$ is analytic in the limit $r \to \infty$, then what constraint needs to be placed on the behaviour of $f(z)$ for the integral to vanish in the limit $R \to \infty$? [3]

(ii) Determine how the value of the integrand just below the branch cut at some $z = x - i\epsilon$ is related to the integrand just above the branch cut at $z = x + i\epsilon$ in the limit $\epsilon \to 0$. [2]

(b) Consider now the function

$$g(z) = \frac{z(z^2+3)}{(z^2+1)(z^2+4)\sqrt{z^2-1}}.$$

(i) Identify the point(s) or region(s) in the complex plane where $g(z)$ is not analytic, stating the nature of the features identified. [3]

(ii) Evaluate the integral

$$J = \int_1^\infty g(x)dx$$

using contour integration around a closed contour. Identify contributions from different parts of the contour. [12]

5A

The Fourier transform of $y(t)$ is given by

$$\tilde{y}(\omega) = \frac{-\omega \tilde{f}(\omega)}{\omega^3 - i\omega^2 + 4\omega - 4i}, \qquad (*)$$

where $\tilde{f}(\omega)$ is the Fourier transform of the function $f(t)$, and both $y(t)$ and $f(t)$ vanish as $t \to \pm\infty$.

(a) Determine the third order differential equation that governs $y(t)$. [3]

(b) Find $f(t)$, valid for all t, for the case

$$\tilde{f}(\omega) = \frac{-i}{\omega - i}. \qquad (\dagger)$$

[5]

(c) Substitute $(\dagger)$ into $(*)$ and use an inverse Fourier transform to determine $y(t)$, valid for all t. Sketch the behaviour of $y(t)$. [12]

6B

(a) Define an order two tensor. [2]

(b) The quantity C_{ij} has the property that for every order two tensor A_{ij}, the quantity $C_{ij}A_{ij}$ is a scalar. Prove that C_{ij} is necessarily an order two tensor. [4]

(c) Show that if a tensor T_{ij} is invariant under a rotation of $\pi/2$ about the x_3-axis then it has the form

$$\begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

Also show that T_{ij} is invariant under a general rotation about the x_3-axis. [6]

(d) The inertia tensor about the origin of a rigid body occupying volume V with mass density $\rho(\mathbf{x})$ is defined as

$$I_{ij} = \int_V \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j)dV.$$

The rigid body B has uniform density ρ and occupies the cylinder

$$\{(x_1, x_2, x_3) : -2 \leqslant x_3 \leqslant 2, \quad x_1^2 + x_2^2 \leqslant 1\}.$$

Show that the inertia tensor of B about the origin is diagonal in the (x_1, x_2, x_3) coordinate system and calculate its diagonal elements. [8]

7A

A mass m_1 is suspended from the origin by a spring with spring constant k_1. A second mass m_2 is suspended from the first by a spring with spring constant k_2. Both springs are of a type that has zero length when not extended. The motion of the masses is restricted to the (x, y) plane such that m_1 is located at $(X_1(t), Y_1(t))$ and m_2 is located at $(X_2(t), Y_2(t))$. Gravity acts in the $-y$ direction.

(a) Write down the Lagrangian for the system and hence use the Euler-Lagrange equation to determine the equations of motion for the system. [6]

(b) Determine the equilibrium position $(\hat{X}_1, \hat{Y}_1, \hat{X}_2, \hat{Y}_2)$. Suppose the setup is altered so that $X_2(t) = \hat{X}_2$. Show that small perturbations $(x_1(t), y_1(t), y_2(t))$ about the equilibrium are governed by

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & 0 & 0 \\ 0 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

[3]

(c) Determine the frequency and structure of each of the modes for the case $m_1 = m_2 = m$ and $k_1 = k_2 = k$. [11]

8C

(a) Given a finite group G of order $|G|$ and a normal subgroup N of order $|N|$, define the quotient group G/N and show that it is indeed a group. State Lagrange's theorem relating the order of a group and those of its subgroups. [2]

(b) Show that the Pauli matrices together with the identity matrix

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

do **not** constitute a group under matrix multiplication. Show that these matrices can be multiplied by ± 1 and $\pm i$, to generate a set of 16 matrices, which meet the conditions to form a group. [9]

(c) Prove that in any group, an element and its inverse have the same order. [9]

9C

(a) Let G be a finite subgroup. The centre $Z(G)$ of G is the set of elements $z \in G$ that commute with every element $g \in G$, that is to say

$$Z(G) = \{z \in G : gz = zg, \forall g \in G\}\,.$$

Prove that if H is a normal subgroup of G with order $|H| = 2$, then $H \subseteq Z(G)$. [7]

(b) Define a *homomorphism* between two groups H and G. Define the *kernel* of a homomorphism. [3]

(c) Suppose that G is a group of order $|G| = 21$. Show that every proper subgroup of G is cyclic. [10]

10A

Let $G = \{I, g_1, g_2, \ldots, g_{n-1}\}$ be a group with a faithful representation by multiplication of 2×2 real orthogonal matrices of the form

$$\mathbf{D}(g_i) = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$$

in the vector space $\mathbb{R}^2$. Suppose $A = \{I, a\}$ and $B_p = \{I, g_1, g_2, \ldots, g_{p-1}\}$ are cyclic subgroups of G such that $g_i = g_1^i$ for $i < p$ for some $2 < p < n$.

(a) Show that $\mathbf{D}(a)$ is symmetric. Obtain relationships between α, β, γ and δ and hence determine the most general form(s) for $\mathbf{D}(a)$. What geometric operations do these form(s) correspond to in the vector space $\mathbb{R}^2$? [*Hint: consider* $\alpha = \cos\theta$.] [5]

(b) What restriction must be placed on p for B_p to have a cyclic subgroup $C = \{I, c\}$? Give the representation $\mathbf{D}(c)$ and hence the representation $\mathbf{D}(g_i)$ for $i < p$. What is the character of B_p for this representation? [6]

(c) Suppose G has generators $\{g_1, s\}$ where $\det(\mathbf{D}(s)) = -1$ and $\{I, s\}$ is a subgroup of G, and B_p is of the form given in (b). What is the order of G? How many cyclic subgroups of order 2 does G have? For the case $p = 4$, give suitable representations for $\mathbf{D}(g_1)$ and $\mathbf{D}(s)$, and use these representations to demonstrate that $sg_1s = g_3$. Determine the group table (for $p = 4$) and identify this group. [9]

END OF PAPER