UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday, 27 May, 2014 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A **separate** *green master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

(i) Using Cartesian coordinates show that

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}\,,$$

and that

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}\,,$$

where $\mathbf{u}$ and $\mathbf{v}$ are three-dimensional vector fields. [6]

(ii) State the divergence theorem and use it to show that

$$\int_V [\mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})]\, dV = \int_S (\mathbf{F} \times \mathbf{G}) \cdot \widehat{\mathbf{n}}\, dS\,,$$

where $\mathbf{F}$ and $\mathbf{G}$ are three-dimensional vector fields, V is a given volume with surface S, and $\widehat{\mathbf{n}}$ is the outward unit vector normal to S. [6]

(iii) Let V be the volume bounded by the plane $z = 0$ and the paraboloid $z = 4 - x^2 - y^2$ with surface S and outward unit normal vector $\widehat{\mathbf{n}}$. If

$$\mathbf{F} = (xz\sin(yz) + x^3, \cos(yz), 3zy^2 - e^{x^2+y^2})\,,$$

find $\int_S \mathbf{F} \cdot \widehat{\mathbf{n}}\; dS$. [8]

2C

The velocity, $u(x,t)$, of a viscous fluid satisfies

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}\,, \qquad (\star)$$

where ν is a positive constant.

(i) Consider the flow of a semi-infinite viscous fluid above a flat oscillating plate with boundary conditions $u(0,t) = U_0 \cos(\omega t)$ and $\lim_{x\to\infty} u(x,t) = 0$. Using the method of separation of variables, solve $(\star)$ for $u(x,t)$. [10]

[*Hint: Consider the complex velocity, v, such that $u = \Re(v)$ where $\Re$ denotes the real part.*]

(ii) A viscous fluid satisfying $(\star)$ is confined between two stationary parallel plates, separated by a distance L. At $t = 0$, the fluid velocity is

$$u(x,0) = U_0 \left(\frac{x}{L} - \frac{x^2}{L^2} \right),$$

and the fluid remains at rest at each plate with boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ for $t \geqslant 0$. Using the method of separation of variables, find a series solution for the velocity $u(x,t)$ for $t \geqslant 0$. Write down an expression for the series coefficients. What is the velocity in the limit as $t \to \infty$? [10]

3B

A beam lies along the x-axis with its ends at $x = 0$ and $x = 1$. The transverse displacement $y(x)$ of the beam when a force per length $f(x)$ is applied satisfies

$$\frac{d^4y}{dx^4} = f(x)\,.$$

The boundary conditions are $y = 0$ and $dy/dx = 0$ at both $x = 0$ and $x = 1$. The displacement can be written in terms of a Green's function $G(x, \xi)$ as

$$y(x) = \int_0^1 G(x, \xi) f(\xi)\, d\xi\,.$$

What conditions must the Green's function satisfy at $x = 0$ and $x = 1$ and at $x = \xi$? [4]

Construct the Green's function to show that

$$G(x,\xi) = \begin{cases} -\frac{1}{6}x^2(\xi-1)^2(x+2x\xi-3\xi) & \text{for } x < \xi\,, \\ -\frac{1}{6}\xi^2(x-1)^2(\xi+2x\xi-3x) & \text{for } x > \xi\,. \end{cases}$$

[12]

Consider two points x_1 and x_2 along the beam. A force $f(x) = \delta(x - x_1)$ causes a displacement $y_1(x_2)$ at x_2. If the force is instead $f(x) = \delta(x - x_2)$, the displacement at x_1 is $y_2(x_1)$. Show that $y_1(x_2) = y_2(x_1)$. [4]

4C

(i) The Fourier transform of a function $f(x)$ is given by

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx\,.$$

Write down the corresponding expression for the inverse Fourier transform. [2]

(ii) Let $g(x) = x^n f(x)$ where n is a positive integer. Derive an expression for $\widetilde{g}(k)$, written in terms of derivatives of $\widetilde{f}(k)$ with respect to k. [4]

(iii) Using the result from part (ii), or otherwise, find the Fourier transform of the following function:

$$f(x) = xe^{-x^2}. \qquad (\star)$$

[*Hint:* $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.] [6]

(iv) Derive Parseval's theorem:

$$\int_{-\infty}^{\infty} [f(x)]^* g(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\widetilde{f}(k)]^* \widetilde{g}(k)dk\,.$$

[4]

(v) The energy, E, of a function $f(x)$ is defined as

$$E = \int_{-\infty}^{\infty} |f(x)|^2 dx\,.$$

Find the energy of the function defined in $(\star)$, and verify that the result is consistent with Parseval's theorem. [4]

5A

(i) Define a Hermitian matrix and show that its eigenvalues are real. Define a unitary matrix and show that its eigenvalues have unit modulus. [7]

(ii) Consider two $n \times n$ matrices U and H that are related by

$$\mathsf{U} = e^{i\mathsf{H}} \equiv \sum_{m=0}^{\infty} \frac{(i\mathsf{H})^m}{m!} \ .$$

If H is Hermitian, show that U is unitary. [5]

(iii) Suppose that a $n \times n$ unitary matrix can be written as $\mathsf{U} = \mathsf{M} + i\mathsf{N}$, where M and N are Hermitian matrices. You may assume that M and N have n distinct eigenvalues.

Show that M and N have the same eigenvectors and determine the eigenvalues of M and N in terms of the eigenvalues of U. [8]

6A

(i) Let M be a $n \times n$ real symmetric matrix. Explain how to construct an orthogonal matrix O such that $\mathsf{O}^{\mathrm{T}}\mathsf{M}\mathsf{O} = \mathsf{D}$, where D is a real diagonal matrix. [4]

(ii) The quadratic form associated with a 3×3 real symmetric matrix M is

$$Q(\mathsf{x}) \equiv \mathsf{x}^{\mathrm{T}}\mathsf{M}\mathsf{x} = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i M_{ij} x_j \ ,$$

where $\mathsf{x}^{\mathrm{T}} = \left[x_1 \, , \, x_2 \, , \, x_3\right]$.

Let Σ be the surface in $\mathbb{R}^3$ defined by

$$Q(\mathsf{x}) = k = \text{const.} \qquad (\star)$$

Define the change of coordinates that brings $(\star)$ into the form [2]

$$\lambda_1 x_1'^2 + \lambda_2 x_2'^2 + \lambda_3 x_3'^2 = k \ .$$

For $k > 0$, describe Σ for the following cases: [4]

(a) $\lambda_1 = \lambda_2 = \lambda_3 > 0$;

(b) $\lambda_1 = \lambda_2 > 0$, $\lambda_3 < 0$;

(c) $\lambda_1 = 0$, $\lambda_2 > 0$, $\lambda_3 > 0$.

(iii) Consider the quadratic surface Σ defined by

$$x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 = 3 \ .$$

Show that Σ has an axis of rotational symmetry and find its direction. [10]

7B

(i) Derive the Cauchy–Riemannn conditions satisfied by the real part $u(x, y)$ and the imaginary part $v(x, y)$ of an analytic function $f(z)$ of the complex variable $z = x+iy$, and show that u and v each satisfy Laplace's equation in two dimensions, i.e., $\nabla^2 u = 0$ and $\nabla^2 v = 0$. [4]

(ii) Show that the equation

$$\left|\frac{z-a}{z+a}\right| = \lambda$$

defines a family of circles in the complex plane and find their centres and radii in terms of the real and positive parameters a and λ. [6]

(iii) A real function $V(x, y)$ satisfies $\nabla^2 V = 0$ in two dimensions in the half-plane $x > 0$ outside a circle of radius R centred on $x = d$ and $y = 0$ (with $d > R$). The function takes values $V = 0$ on $x = 0$ and $V = -V_0$ on the circle. By considering the real part of the complex function

$$f(z) = \ln\left(\frac{z-a}{z+a}\right) ,$$

or otherwise, show that

$$V = \frac{V_0}{\cosh^{-1}(d/R)} \ln\left|\frac{z-a}{z+a}\right| ,$$

for a suitable constant a that should be determined. [10]

8B

Consider the ordinary differential equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \left[x^2 - l(l+1)\right] y = 0\,,$$

where l is a non-negative integer. Find and classify the singular points of the equation. [4]

The differential equation admits two linearly-independent solutions of the form

$$y(x) = x^\sigma \sum_{n=0}^{\infty} a_n x^n\,, \qquad (a_0 \neq 0)\,.$$

Determine the two possible values of σ and the recursion relations satisfied by the a_n in each case. [10]

Using these recursion relations, verify that, for a suitable choice of a_0, the solution that is regular at $x = 0$ is

$$y(x) = 2^l x^l \sum_{s=0}^{\infty} \frac{(-1)^s (s+l)!}{s!(2s+2l+1)!} x^{2s}\,.$$

Express this series for $l = 0$ in terms of elementary functions and verify directly that your result satisfies the differential equation. [6]

9A

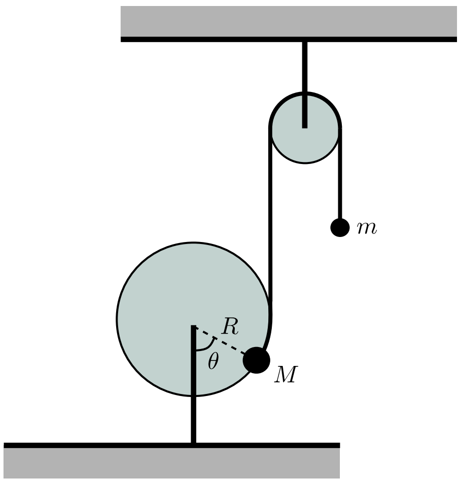
(i) Derive the Euler–Lagrange equation for the function $q(t)$ corresponding to stationary values of the functional

$$S[q(t)] = \int_{t_0}^{t_1} L(t, q(t), \dot{q}(t))\, dt \ , \qquad \dot{q} \equiv dq/dt \ ,$$

for fixed $q(t_0)$ and $q(t_1)$. [5]

What is the first integral of the Euler–Lagrange equation if L is independent of t? [5]

(ii) A mass M is attached to a massless hoop of radius R. The hoop lies in a vertical plane and is free to rotate about its fixed center. A massless, inextensible string connects M to a second mass $m < M$ as shown in the figure (i.e., the string winds part way around the hoop, then rises vertically up and over a massless pulley). Assume that m moves only vertically in a uniform gravitational field (with gravitational acceleration g). You may ignore friction.



The Lagrangian L is the difference of the kinetic and potential energies of the system. From the Euler–Lagrange equation find the equation of motion for the angle of rotation of the hoop, $0 \leqslant \theta(t) \leqslant \pi/2$.

Derive the equilibrium angle θ_0. Consider small oscillations around θ_0, i.e., let $\theta(t) = \theta_0 + \delta(t)$, where $|\delta| \ll \theta_0$. Show that the angular frequency of oscillations is

$$\omega = \left(\frac{M-m}{M+m}\right)^{1/4} \sqrt{\frac{g}{R}} \ .$$

Comment on the limit $M \gg m$. [10]

10A

(i) The Sturm–Liouville equation is

$$-\left[p(x)\psi'\right]' + q(x)\psi = \lambda w(x)\psi \ , \qquad (\star)$$

where $p(x) > 0$ and $w(x) > 0$ for $a \leqslant x \leqslant b$, and primes denote differentiation with respect to x. Show that finding the eigenvalues λ is equivalent to finding the stationary values of the functional

$$\Lambda[\psi(x)] = \frac{\int_a^b (p\psi'^2 + q\psi^2)dx}{\int_a^b w\psi^2 \, dx} \ ,$$

if suitable boundary conditions are satisfied at $x = a$ and $x = b$ (which should be stated). [6]

Let λ_0 be the lowest eigenvalue and ψ_0 be the associated eigenfunction. A general function $\tilde{\psi}$ can be written as

$$\tilde{\psi}(x) = c_0\psi_0(x) + \sum_{i=1}^{\infty} c_i\psi_i(x) \ ,$$

where c_0 and c_i are constants, and ψ_i $(i = 0, 1, 2, \ldots)$ are orthonormal eigenfunctions of $(\star)$ with eigenvalues $\lambda_i \geqslant \lambda_0$. Show that

$$\tilde{\lambda} \equiv \Lambda[\tilde{\psi}(x)] = \frac{\lambda_0 + \sum_{i=1}^{\infty} |a_i|^2\lambda_i}{1 + \sum_{i=1}^{\infty} |a_i|^2} \ ,$$

where $a_i \equiv c_i/c_0$. Explain how this result allows you to estimate the lowest eigenvalue λ_0. [6]

(ii) Consider the Schrödinger equation

$$-\psi'' + x^2\psi = \lambda\psi \ ,$$

for $0 \leqslant x < \infty$ and with the boundary conditions $\psi(0) = 0$, $\lim_{x\to\infty} \psi(x) = 0$. Using the trial function $\tilde{\psi} = xe^{-\alpha x}$ with α a real positive constant, estimate the lowest eigenvalue λ_0. [8]

END OF PAPER