UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 9 June, 2014 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

*You may submit answers to **all** of section A, and to no more than **five** questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

After the end of the examination:

*Tie up **all of your section A answer** in a single bundle, with a completed blue cover sheet.*

*Each section B question has a number and a letter (for example, **11X**). Answers to each question must be tied up in **separate** bundles and marked (for example **11X, 12T** etc) according to the number and letter affixed to each question. **Do not join the bundles together**. For each bundle, a blue cover sheet **must** be completed and attached to the bundle, with the correct number and letter written in the box.*

*A **separate** green master cover sheet listing all the questions attempted **must** also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.
Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS	***SPECIAL REQUIREMENTS***
6 blue cover sheets and treasury tags	*None*
Green master cover sheet	
Script paper	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(a) Differentiate

$$\frac{1}{x^2+4}$$

with respect to x. [1]

(b) Differentiate

$$e^{\sin x}$$

with respect to x. [1]

2

(a) Differentiate a^{-x} with respect to x, where a is a constant which satisfies $a > 0$ and $a \neq 1$. [1]

(b) Evaluate the indefinite integral

$$\int \frac{\ln(\ln x)dx}{x}.$$

[1]

3

(a) Evaluate the definite integral

$$\int_0^{\frac{\pi}{4}} \tan x \; dx.$$

[1]

(b) Evaluate the definite integral

$$\int_{-2}^{-1} \frac{dx}{x}.$$

[1]

4

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = \cos^2 y \sin x$$

for $-\pi/2 < y < \pi/2$. [1]

(b) Find the solution of the differential equation

$$\frac{dy}{dx} = 3y$$

such that $y = 3$ when $x = 0$. [1]

5

(a) Find all pairs of coordinates where the curve defined by

$$7x^2 - y^2 = 7$$

meets the straight line

$$y = x + 1.$$

[1]

(b) Sketch the curve defined by the equation $(x-1)^2 + 2y^2 = 3$. [1]

6

(a) A sphere has radius 10 m. Drawn on its surface is a circular patch which has area 10 m^2. What fraction of the surface of the sphere does this cover? [1]

(b) A square pyramid has all of its sides of length 1 m. What is its volume? [1]

7

(a) Sketch the graph of

$$y = \frac{1}{1 + \tan x}.$$

[1]

(b) Sketch the graph of

$$y = e^{-x^3}.$$

[1]

8

(a) Find the values of x at the stationary points of the function

$$y = x^3 - 2x^2 - 7x + 6.$$

[1]

(b) The function $y = x^3 - 3x + 7$ has a stationary point at $x = 1$. Is this point a maximum, minimum or a point of inflection? [1]

9

(a) What is the area bounded by the curve $y = x^2 - 3x + 2$, the positive half of the x-axis, the positive half of the y-axis and the line $x = 1/2$? [1]

(b) Sketch the curve(s) defined by the relation

$$y^2 = x^3.$$

[1]

10

(a) The straight line L is defined by the equation $y = 2x + 3$. Find the equation of the line L' that is perpendicular to L and intersects L where L crosses the x-axis. [1]

(b) Express

$$\frac{13(x+1)}{(x-4)(x+9)}$$

as a sum of partial fractions. [1]

SECTION B

11X

(a) Calculate Det $\mathbf{A}$ and Tr $\mathbf{A}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix}.$$

From the value of Det $\mathbf{A}$ make a deduction about the eigenvalues of $\mathbf{A}$. [4]

(b) Calculate the eigenvalues and the corresponding normalised eigenvectors of $\mathbf{A}$. Verify that the eigenvectors are mutually orthogonal. [10]

(c) By expressing an arbitrary vector $\mathbf{r}$ in terms of the eigenvectors or otherwise, show that a non-zero vector $\mathbf{e}$ exists such that

$$\mathbf{Ar} \cdot \mathbf{e} = 0$$

for all $\mathbf{r}$. [4]

(d) Describe in words the action of $\mathbf{A}$ on an arbitrary non-zero vector. [2]

12T

(a) Express the cube roots of $i - 1$ in terms of their modulus and argument. [4]

(b) Find all the solutions to the equation

$$\tanh z = -i.$$

[5]

(c) Given that $z = 2 + i$ solves the equation

$$z^3 - (4+2i)z^2 + (4+5i)z - (1+3i) = 0\,,$$

find the remaining solutions. [6]

(d) Use complex numbers to show that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

[5]

13Y

(a) Solve the following differential equations for $y(x)$ subject to the listed boundary conditions, making your answer explicit for y.

(i) $\dfrac{dy}{dx} + 3y = 8$, with $y(0) = 4$. [3]

(ii) $\dfrac{dy}{dx} - y\cos x = \dfrac{1}{2}\sin 2x$, with $y(0) = 0$. [7]

(b) The function $y(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 2e^{-3x}\,.$$

Solve the equation for $y(x)$ subject to $y(0) = 1$ and $\left.\dfrac{dy}{dx}\right|_{x=0} = 0$. [10]

14Z

(a) An arbitrary point along a straight line in a three-dimensional space can be written as $\mathbf{r}_1 = \mathbf{a} + \lambda\hat{\mathbf{b}}$, where λ is a scalar parameter and $\hat{\mathbf{b}}$ is a unit vector. Obtain a formula for the minimum distance between $\mathbf{r}_1$ and $\mathbf{r}_2 = \mathbf{c} + \mu\hat{\mathbf{d}}$, where $\hat{\mathbf{d}}$ is a unit vector, assuming that the two lines are not parallel. [5]

(b) Find all vectors $\mathbf{x}$ that obey the equation $\mathbf{x}\cdot\mathbf{p} = k$, where $\mathbf{p}$ is a fixed non-zero vector in a three-dimensional space and k is a fixed scalar.

[*Hint: Your answer should contain an arbitrary non-zero vector* $\mathbf{q}$ *which can be taken to be non-collinear with* $\mathbf{p}$, *i.e.,* $\mathbf{p}\times\mathbf{q} \neq 0$. *You should treat the cases* $\mathbf{p}\cdot\mathbf{q} \neq 0$ *and* $\mathbf{p}\cdot\mathbf{q} = 0$ *separately.*] [7]

[*Notation:* $\mathbf{u}\times\mathbf{v}$ *is equivalent to* $\mathbf{u}\wedge\mathbf{v}$.]

(c) A particle moves along a path on which the position coordinate in terms of a parameter t is given by

$$x = \frac{\cos t}{\sqrt{1+t^2}}, \quad y = \frac{\sin t}{\sqrt{1+t^2}}, \quad z = \frac{t}{\sqrt{1+t^2}}.$$

Express the equation for a point on the path in spherical polar coordinates $r(t), \theta(t), \phi(t)$. [8]

15W

(a) The area of integration, D, is defined in plane polar coordinates (r, ϕ) by the inequality $r_2 \leqslant r \leqslant r_1$, where $r_1 = 1 + \cos\phi$ and $r_2 = 3/2$.

(i) Sketch the area of integration. [4]

(ii) Calculate the value of the area D. [6]

(iii) Evaluate the following integral over this area:

$$\iint\limits_D \frac{x + y + xy}{x^2 + y^2}\, dxdy\,.$$

[5]

(b) Evaluate the triple integral:

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} x \exp\left[-\frac{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}{a^2}\right]\, dxdydz\ ,$$

where $a > 0$, x_0, y_0 and z_0 are real constants. [5]

16Z

(a) A function f of two variables x and y is defined as

$$f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x.$$

Determine the positions of the stationary points of f and their characters (maximum, minimum or saddle point). [8]

(b) A function g of two variables x and y is defined as

$$g(x, y) = x^4 + y^4 - 36xy.$$

Sketch the contours of g in the x-y plane, indicating on the sketch the positions and characters of all the stationary points. [12]

17S

(a) Suppose $f(x)$ is a periodic function with period 2π. Write down its Fourier series and give expressions for the coefficients that appear in it. [3]

(b) The function $g(x) = \mathrm{e}^x$ is defined on the interval $-\pi \leqslant x < \pi$. Sketch the periodic continuation of $g(x)$ with period 2π, between $x = -3\pi$ and $x = 3\pi$. If we were to calculate the Fourier series of this periodic continuation of $g(x)$, what value would it take at the point $x = \pi$? [4]

(c) Consider the function $h(x) = x(\pi - x)$ defined on the interval $0 \leqslant x < \pi$. By considering the appropriate periodic continuation of $h(x)$ over the real line, show that the half-range sine series for $h(x)$ is

$$h(x) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\left[(2n+1)x\right].$$

[10]

Hence demonstrate that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

[3]

18S

(a) Suppose X is a discrete random variable that takes the integer values $0, 1, 2, \ldots, N$. Its normalised probability distribution is denoted by $P(X)$.

Write down expressions for the mean μ, the variance σ^2, and the probability $P(X < Y)$, where Y is a fixed positive integer less than N. [3]

(b) A pond contains K trout and $N - K$ carp. Bruce goes fishing at the pond and in one day catches M fish. Note that Bruce never returns a fish to the pond once it is caught.

Calculate the number of ways M fish (of any species) can be caught from a pond of N fish. Suppose Bruce catches X trout out of his haul of M fish. Show that the number of ways of catching X trout out of the haul of M fish is

$$\binom{K}{X}\binom{N-K}{M-X}.$$

Assuming that trout and carp are equally likely to be caught, show that the probability that Bruce catches X trout in one day is

$$P(X) = \binom{K}{X}\binom{N-K}{M-X} \Big/ \binom{N}{M}.$$

[5]

(c) Suppose the pond contains 2 trout and 8 carp, and Bruce catches 2 fish in total. What is the probability that of these two fish (i) none are trout, (ii) one is a trout, and (iii) both are trout? Verify that the three probabilities sum to 1. Consequently, determine the mean and variance of the probability distribution $P(X)$ for this case. (You may leave your answers in reduced fractional form.) [10]

[*Recall that* $\binom{N}{n} = N!/(n!(N-n)!)$.]

(d) The next day Bruce goes fishing at a large lake that contains only trout and carp but in enormous quantities, i.e. both K and $N - K$ are much larger than M. In this limit $P(X)$ approaches the binomial distribution. Give a qualitative explanation for why this is so. [2]

19Y*

(a) A differentiable function $f(x)$ is expanded using the Maclaurin series. Derive an expression which determines the interval for x within which the series is absolutely convergent. [3]

Use your result to determine the interval for x within which the Maclaurin series for $f(x) = \ln(2 + x)$ converges absolutely. Determine whether or not the series converges at the end points of the interval. [4]

(b) Find the first three terms in the Maclaurin series for

$$f(x) = e^{-x}(1 + x)^{-1/2} \ .$$

[3]

(c) Establish the convergence or divergence of the series $\sum_{n=1}^{\infty} u_n$ whose n-th terms, u_n, are

(i) $\dfrac{2^n}{n \ln n}$, [2]

(ii) $\dfrac{\sin n}{n^2}$. [2]

(d) Sum the series

$$S(x) = \frac{x^4}{3(0!)} + \frac{x^5}{4(1!)} + \frac{x^6}{5(2!)} + \ldots .$$

[6]

20R*

The point (a, b) is a stationary point of the function $f(x, y)$ subject to the constraint $g(x, y) = 0$. Using the method of Lagrange multipliers show that

$$\begin{vmatrix} \dfrac{\partial f}{\partial x}(a,b) & \dfrac{\partial g}{\partial x}(a,b) \\[2ex] \dfrac{\partial f}{\partial y}(a,b) & \dfrac{\partial g}{\partial y}(a,b) \end{vmatrix} = 0\,.$$

[4]

(a) By considering the function $f(x, y) = x^2 + y^2$, use the method of Lagrange multipliers to find the maximum distance from the origin to the curve

$$x^2 + y^2 + xy - 4 = 0\,.$$

[4]

(b) In a school, two horizontal corridors, $0 \leqslant x \leqslant a,\ y \geqslant 0$ and $x \geqslant 0,\ 0 \leqslant y \leqslant b$ meet at right angles. The caretaker wishes to know the maximum possible length, L, of a ladder that may be carried horizontally around the corner. Regarding the ladder as a stick, use the method of Lagrange multipliers to calculate L by first placing the ends of the ladder at the points $(a + X, 0)$ and $(0, b + Y)$ and imposing the condition that the corner (a, b) be on the ladder. Then show that at the constrained stationary point, the value of X satisfies the equation

$$(X^3 - ab^2)(X + a) = 0\,,$$

and hence show that $L = (a^{2/3} + b^{2/3})^{3/2}$. [12]

END OF PAPER