UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 10 June, 2013 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11W****). Answers to each question must be tied up in* ***separate*** *bundles and marked (for example* ***11W, 12Z*** *etc) according to the number and letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct number and letter written in the box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.
Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Consider $y(x)$ defined implicitly by $2x^2 + 3y^2 - 6 = 0$.

(a) Differentiate this equation with respect to x. [1]

(b) Find the maximal value of y. [1]

2

(a) Find the coordinates of the point of intersection of the parabolas $y = x^2$ and $y = (x-1)^2$. [1]

(b) Find the angle between the tangents to these parabolas at the point of intersection. [1]

3

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = e^{x-y}.$$

[1]

(b) Find the solution for which $y = \ln 2$ when $x = 0$. [1]

4

Find the gradient and the intercept with the y-axis of the normal to the curve $f(x) = x^2 - 1$ at the point $x = 3$. [2]

5

(a) Sketch the function $f(x) = x\ \tan x$ for $0 \leqslant x \leqslant 2\pi$, $x \neq \pi/2,\ 3\pi/2$. [1]

(b) Find the number of solutions of

$$\begin{cases} y = x \tan x \\ x^2 + y^2 = 25, \end{cases}$$

for which $0 \leqslant x \leqslant 2\pi$. [1]

6

Evaluate the indefinite integral

$$I = \int x^m \ln x\ dx$$

(a) for $m \neq -1$, [1]

(b) for $m = -1$. [1]

7

(a) Express

$$\frac{7x + 29}{(x-3)(x+7)}$$

as a sum of partial fractions. [1]

(b) Complete the square for $x^2 - 6x + 14$. [1]

8

Differentiate the following with respect to x

(a) a^x, where a is a positive constant. [1]

(b) $\sin(\cos x)$. [1]

9

One side of a rectangle has fixed length $a = 10\,\text{cm}$ while the length of the other side, b, increases at a rate of $4\,\text{cm/s}$.

(a) Write the expression for the length of the diagonal of the rectangle as a function of time. [1]

(b) Calculate the rate of change of the length of the diagonal when $b = 30\,\text{cm}$. [1]

10

(a) Sketch the curves $y = x^2 - 1$ and $y = 1 - x^4$. [1]

(b) Evaluate the area bounded by these two curves. [1]

SECTION B

11W

(a) Write the following set of simultaneous equations,

$$\begin{cases} x + 2y + 3z &= 14 \\ x - 2y + z &= 0 \\ 2x + y - z &= 1 \ , \end{cases}$$

in matrix form $\boldsymbol{Ax} = \boldsymbol{b}$, where

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

[2]

Solve this set by

(i) Gaussian elimination, [6]

(ii) Cramer's rule or by inverting matrix $\boldsymbol{A}$. [6]

(b) Show graphically the locus of points in the a-b plane for which the following set of simultaneous equations has non-trivial solutions for x and y,

$$\begin{cases} x\ \sqrt{a^2 - 2ab + b^2} + y\ \sin\left(\pi\sqrt{a^2+b^2}\right) &= 0 \\ x\ \sqrt{a^2 + 2ab + b^2} + y\ \sin\left(\pi\sqrt{a^2+b^2}\right) &= 0 \ , \end{cases}$$

where a and b are real. [6]

12Z

(a) Determine the cube roots of 8, giving your answer in the form $x + iy$. [3]

(b) Find the solutions of $z^2 - (3 + i)z + (2 + i) = 0$. [4]

(c) Consider the polynomial $p(z) = 2z^4 + az^3 + bz^2 + c$, where a, b and c are real.

Given that two roots of $p(z)$ are 2 and i, what are the values of the other two roots? [6]

(d) A particle moves in the x, y plane such that its position as a function of time is given by the real and imaginary parts of the complex number $z(t)$, where

$$z(t) = \frac{2t + i}{t - i}.$$

Determine the magnitudes of the particle's velocity $\dfrac{dz}{dt}$ and acceleration $\dfrac{d^2z}{dt^2}$ as functions of t. [7]

13Z

(a) Find the solution of

$$\frac{dy}{dx} + \frac{4x}{1 + x^2}\, y = \frac{1}{(1 + x^2)^3}$$

that obeys $y = 1$ when $x = 0$. [6]

(b) Find the general solution of

$$\frac{dy}{dx} = \frac{y}{x + y + 2}.$$

[7]

(c) Find the general solution of

$$\frac{dy}{dx} + \frac{2}{x}\, y = -x^2 y^2 \cos x.$$

[7]

14T

(a) Given the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ show that

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 \, |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 \, .$$

[5]

(b) Derive the following expression for the distance between a point at position $\boldsymbol{r}_0$ and the straight line that passes through the two points at positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$:

$$D = \frac{|(\boldsymbol{r}_1 - \boldsymbol{r}_0) \times (\boldsymbol{r}_2 - \boldsymbol{r}_0)|}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}.$$

[5]

(c) Write down the equation of the plane that contains the point $(4, 2, 1)$ and the straight line that passes through the points $(0, 1, 1)$ and $(2, 1, 3)$. [5]

(d) Let P, Q and R be the points of intersection of this plane with the X, Y and Z axes, respectively. Calculate the area of the triangle PQR. [5]

15R

Consider the paraboloid defined in Cartesian coordinates (x, y, z) by the equation $az = x^2 + y^2$, with $a > 0$.

(a) Write the equation of the paraboloid in cylindrical polar coordinates (r, θ, z). [3]

(b) Find the surface area of the paraboloid limited by the planes $z = 0$ and $z = h$. [4]

(c) A cup in the shape of the paraboloid is filled with a liquid up to height $z = h$. The liquid has constant density $\rho(\boldsymbol{r}) = \rho_0$.

(i) Find the volume V occupied by the liquid. [6]

(ii) Find the Cartesian coordinates of the centre of mass of the liquid.
[The centre of mass of a body is the point $\boldsymbol{R}$ defined by

$$\boldsymbol{R} = \frac{1}{M} \iiint \boldsymbol{r} \rho(\boldsymbol{r}) dV,$$

where M is the total mass and $\rho(\boldsymbol{r})$ is the density.] [7]

16Y

(a) Identify, and classify as maxima, minima or saddle points, the stationary points of the function

$$f(x, y) = xy(x^2 + y^2 - 1).$$

[5]

Sketch the contours of $f(x, y)$. [3]

(b) (i) Show that $y(x, t) = F(2x + 5t) + G(2x - 5t)$ is a general solution of

$$4\frac{\partial^2 y}{\partial t^2} = 25\frac{\partial^2 y}{\partial x^2},$$

where F and G are arbitrary differentiable functions. [6]

(ii) Find a solution which satisfies the conditions

$$y(0, t) = y(\pi, t) = 0, \quad y(x, 0) = \sin 2x \quad \text{and} \quad \left.\frac{\partial y}{\partial t}\right|_{(x,0)} = 0.$$

[6]

17S

(a) Given a real function $f(x)$ with period $2L$, write $f(x)$ in terms of a Fourier series, and give the Fourier coefficients. [2]

Using the orthogonality relations or otherwise, determine

$$\int_{-L}^{L} [f(x)]^2 \, dx$$

in terms of squares of the Fourier coefficients (Parseval's Theorem). [6]

(b) Sketch the function

$$f(x) = \begin{cases} x + \pi, & -\pi < x < 0, \\ 0, & 0 < x < \pi, \end{cases}$$

[1]

and find its Fourier series. [11]

18S

(a) From a group of 12 people, comprising six women and six men, six people are selected at random. What is the probability that three women and three men are selected? [4]

(b) Old Street School used three different companies to purchase their supply of light bulbs. Company A provided 20% of the bulbs, Company B provided 40% of the bulbs and Company C provided the rest of the bulbs. It is known that fraction α of the bulbs provided by Company A are defective, that fraction $\alpha/2$ of the bulbs provided by Company B are defective, and that fraction $\alpha/4$ of the bulbs provided by Company C are defective (where $0 \leqslant \alpha \leqslant 1$). Clearly stating the events and their probabilities, answer the following:

(i) What is the probability that a defective bulb was provided by Company A? [7]

(ii) What is the probability that a randomly selected bulb is not defective? [3]

(iii) What is the probability that a non-defective bulb was provided by Company C? [6]

19X*

(a) Explain formally what is meant by the statement

$$f(x) = O(x^2) \quad \text{as} \quad x \to 0\,.$$

[3]

(b) Determine which of the three functions f, g, h is greatest and which is least, as $x \to 0$ from above

$$\begin{aligned} f(x) &= 1 - \cos x \\ g(x) &= x - \sin x \\ h(x) &= \sqrt{e^x - 1}. \end{aligned}$$

[6]

(c) State the ratio test for absolute convergence of the series

$$S_N = \sum_{k=1}^{N} u_k.$$

[3]

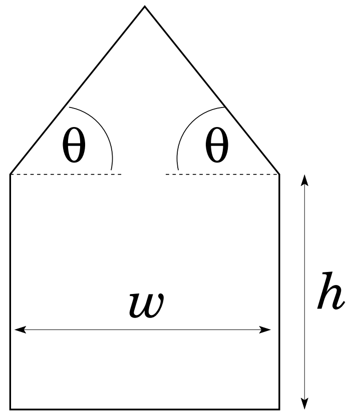
(d) By comparison with the geometric series or otherwise investigate the convergence for different values of p of the series $u_k = k^{-p}$. [8]

20Y*

(a) Use the method of Lagrange multipliers to determine the volume of the largest rectangular parallelepiped (cuboid) which fits inside a hemisphere of radius a.

[A rectangular parallelepiped is a polyhedron with six rectangular faces.] [8]

(b) A warehouse is to be constructed with uniform cross-sectional area A throughout its fixed length. The cross section is to be a rectangle of height h (which is fixed) and width w (which is to be optimised), surmounted by an isosceles triangle roof that makes angles θ with the horizontal, as shown in the figure.



The cost of construction is α per unit height of the walls plus β per unit (slope) length of roof. Show that cost is minimised if w and θ are chosen such that

$$2 \tan 2\theta = w/h$$

and

$$w^4 = 16A(A - wh),$$

irrespective of the values of α and β. [12]

END OF PAPER