UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday, 1 June, 2012 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the left hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B

Define a *scalar product* between two scalar functions $y_1(x)$ and $y_2(x)$ with weight function $w(x)$. [2]

Express the following equation for the function $y(x)$ in Sturm-Liouville form

$$y'' + \left(\frac{1}{x} - 1\right) y' + \frac{n}{x} y = 0 \ ,$$

where $n \geqslant 0$ is an integer. Find the required boundary conditions for the linear operator on $y(x)$ to be self-adjoint over the interval $[0, \infty]$. Show that as long as the eigenfunctions of the operator are polynomials, the boundary conditions are always satisfied. What is the orthogonality condition for this linear operator? [6]

Show that

$$u(x) = -x + 1 \ , \quad \text{and} \quad v(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6) \ ,$$

are eigenfunctions of the linear operator, and find the corresponding eigenvalues. By assuming that the eigenfunction $h(x)$ associated with $n = 2$ is a polynomial of order 2, find $h(x)$ given $h(1) = 1$. [6]

Find two solutions for the case $n = 0$, given the boundary condition $y(x_0) = 1$ for $0 < x_0 < \infty$. You may leave the solutions in integral form. Show that one of the solution diverges at $x = \infty$. [6]

2B

Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ,$$

with $t \geqslant 0$ on the interval $x \in [0, 1]$ with boundary conditions

$$\frac{\partial u}{\partial x}(0, t) = -u(0, t) , \quad \frac{\partial u}{\partial x}(1, t) = -u(1, t) .$$

Using the method of separation of variables $u(x, t) = W(x)T(t)$, with the separation constant k, show that $k = 0$ yields the trivial solution $u(x, t) = 0$. [4]

By separately considering solutions for $k > 0$ and $k < 0$, show that the general solution to the diffusion equation with the given boundary conditions can be written as

$$u(x, t) = c_0 e^{-x} e^{t} + \sum_{n=1}^{n=\infty} c_n e^{-n^2\pi^2 t} W_n(x),$$

where

$$W_n(x) = n\pi \cos(n\pi x) - \sin(n\pi x) ,$$

and c_0 and c_n are constants of integration. [10]

Given the initial condition $u(x, 0) = f(x)$ where $f(x)$ is an arbitrary function, find the coefficients c_0 and c_n as integrals involving $f(x)$. You may assume without proof that the functions W_n are orthogonal to each other and to e^{-x} in the interval $x \in [0, 1]$. [6]

3B

Let $u(\mathbf{r})$ and $v(\mathbf{r})$ be scalar fields that tend to zero as $|\mathbf{r}| \to \infty$, with the coordinates $\mathbf{r} = (x, y, z)$. Use the divergence theorem to show that

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, (u\mathcal{L}_H v - v\mathcal{L}_H u) = 0,$$

where $\mathcal{L}_H = \nabla^2 + k_0^2$ is the Helmholtz operator and $k_0 > 0$ is a real constant. [4]

The eigenfunctions for the following equation

$$\nabla^2 \psi_{\mathbf{k}}(\mathbf{r}) = -k^2 \psi_{\mathbf{k}}(\mathbf{r}) \ ,$$

are given by

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}} \ , \qquad (\star)$$

with eigenvalues $-k^2$ where $k \equiv |\mathbf{k}|$. Show that these eigenfunctions satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \ \psi^*_{\mathbf{k}_1}(\mathbf{r}) \, \psi_{\mathbf{k}_2}(\mathbf{r}) = \delta^3(\mathbf{k}_1 - \mathbf{k}_2) \ .$$

[2]

Consider the Helmholtz equation with a source

$$(\nabla^2 + k_0^2)\Phi(\mathbf{r}) = V(\mathbf{r}) \ .$$

By expanding the solution $\Phi(\mathbf{r})$ using the eigenfunctions $(\star)$,

$$\Phi(\mathbf{r}) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \ A_{\mathbf{k}} \psi_{\mathbf{k}}(\mathbf{r}),$$

where $A_{\mathbf{k}}$ are complex coefficients and (k_x, k_y, k_z) are components of $\mathbf{k}$, show that

$$\Phi(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \ \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k_0^2 - k^2} V(\mathbf{r}').$$

[8]

Hence show that the Green's Function for the Helmholtz operator $\mathcal{L}_H$ is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi^2 l} \int_0^{\infty} \frac{\sin(kl)}{k^2 - k_0^2} \ k \ dk \ ,$$

where $l = |\mathbf{r} - \mathbf{r}'|$. [6]

4B

(i) State the Cauchy integral formula for a function $f(z)$ which is analytic on a closed contour C and within the interior region bounded by C. [2]

Use the Cauchy integral formula to calculate

$$\oint_C \frac{dz}{z^2 - 1} \ ,$$

where C is the circle $|z| = 2$. [4]

(ii) The Laurent expansion of a complex function $f(z)$ about a point z_0 is given by

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n (z - z_0)^n \ .$$

Write the expression for the cofficients a_n as a contour integral, with the contour C within the annular region $r < |z - z_0| < R$ encircling z_0 once in a counterclockwise sense, assuming that such an annular region of convergence of $f(z)$ exists. [2]

For the complex function

$$f(z) = \frac{1}{z(z-1)} \ ,$$

show that the Laurent expansion about $z_0 = 0$ is given by

$$f(z) = -\sum_{n=-1}^{\infty} z^n \ .$$

[12]

5C

The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ is defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx}\; f(x)\; dx\,.$$

Prove the convolution theorem; namely that if

$$\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$$

then

$$h(x) = \int_{-\infty}^{\infty} f(y)\; g(x-y)\; dy\,.$$

[6]

Suppose that $f(x) = e^{-|x|}$. Show that the convolution of $f(x)$ with itself is given by

$$h(x) = \begin{cases} (1-x)e^{x}\,, & x<0 \\ (1+x)e^{-x}\,, & x>0\,. \end{cases}$$

[10]

Hence, use the convolution theorem to show that

$$h(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{(1+k^2)^2}\; dk\,.$$

[4]

6C

Define the terms *tensor* and *isotropic tensor*. [2]

Show that the Kronecker delta δ_{ij} is an isotropic tensor.

Let A and B be a pair of rank two tensors. The determinant of a rank two tensor whose components are A_{ij} is given by

$$\det A = \frac{1}{6}\epsilon_{ijk}\epsilon_{lmn}A_{il}A_{jm}A_{kn}\,.$$

Show that

$$\det(AB) = \det A \det B\,.$$

[10]

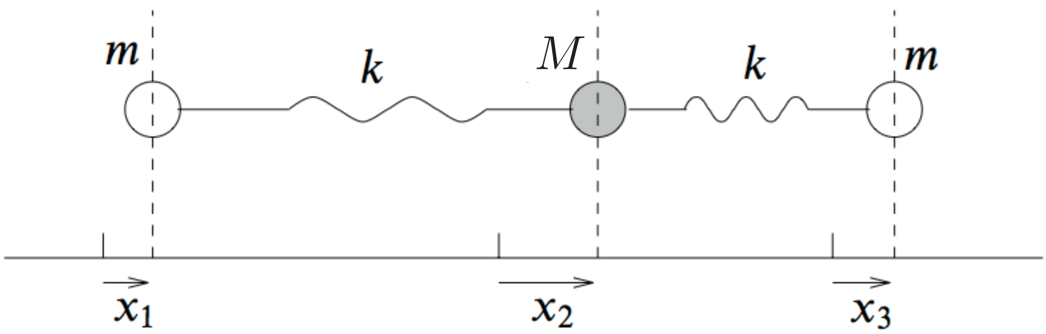
Suppose that A depends on a parameter t and is invertible. Show that

$$\frac{d}{dt}\det A = \left(\mathrm{Tr} A^{-1}\,\frac{dA}{dt}\right)\,\det A\,.$$

[8]

7C

A model of the carbon dioxide molecule is sketched below



where the central atom is carbon which has mass M and the two oxygen atoms have mass m. The vibrational motion in the x-direction can be modelled by springs joining the carbon atom with the oxygen atoms. Let the springs have spring constant k.

For vibrations in the x-direction, find the normal modes and their eigenfrequencies. [6]

Give a brief explanation for the occurrence of any zero modes. [4]

Suppose that initially the molecule is in equilibrium and the left-hand oxygen atom starts to move with speed u in the positive x-direction whilst the other two atoms are stationary. Describe the subsequent motion in terms of the normal modes you have found. [10]

8B

Define the *order* of a finite group G. What is meant by a *normal* subgroup H of G? [2]

Consider D_4, the symmetry group of a square. Identify the elements of this group, and explain their geometrical action on the square. List all 8 proper subgroups. Hence identify the order 2 normal subgroup of D_4. [8]

Consider now the D_n group, the symmetry group of an n-gon for $n \geqslant 3$. Prove that when n is even, there exists only one order 2 normal subgroup while when n is odd, there exists no order 2 normal subgroup. [10]

9B

Define a *homomorphism* and an *isomorphism* between two groups G_1 and G_2. [2]

Let $M(n)$ be the set of all real $n \times n$ matrices.

Show that the subset of M,

$$GL(n) = \{A \in M \text{ such that } \det(A) \neq 0\},$$

forms a group under the usual law of matrix multiplication. [2]

Show that the following two subsets of $GL(n)$

$$SO(n) = \{A \in M \text{ such that } \det(A) = 1 \text{ and } AA^T = \mathbb{I}\},$$

and

$$GL^+(n) = \{A \in M \text{ such that } \det(A) > 0\},$$

are subgroups of $GL(n)$. [4]

Now consider $SO(2)$, the set of all real 2×2 matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

such that $MM^T = \mathbb{I}$ and $\det M = 1$. By finding a suitable parametrization for (a, b, c, d), prove that $SO(2)$ is isomorphic to $U(1)$, the group of all complex numbers of modulus one under the usual multiplication of complex numbers. [12]

10B

Any theorems you use should be stated, but need not be proven in this question.

(i) State the definition of a *conjugacy class* of a group. State the definition of an *irreducible* representation. [2]

Consider the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ with the defining relations

$$-1 = i^2 = j^2 = k^2 = ijk \ .$$

Find all conjugacy classes of Q. Hence deduce the number of irreducible representations of Q and state their dimensions. [8]

(ii) Consider the 3-dimensional real matrix representation T of the order 4 cyclic group $Z_4 = \{I, a, a^2, a^3\}$ given by

$$T(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix} \ .$$

What are the conditions on the real constants b and c such that T is (i) a *faithful* representation, and (ii) an *unfaithful* representation? [7]

Finally, construct a 2-dimensional representation of Z_4 with kernel $\{I, a^2\}$. [3]

END OF PAPER