UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 13 June, 2012 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11X****). Answers to each question must be tied up in* ***separate*** *bundles and marked (for example,* ***11X, 12W*** *etc) according to the number and letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct number and letter written in the section box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Given vectors $\boldsymbol{u} = (a, b, 0)$, $\boldsymbol{v} = (0, 1, c)$ and $\boldsymbol{u} \wedge \boldsymbol{v} = (3, -6, 2)$, determine the constants a, b and c.

[Notation: $\boldsymbol{u} \wedge \boldsymbol{v}$ is equivalent to $\boldsymbol{u} \times \boldsymbol{v}$.] [2]

2

Evaluate the integral

$$\int_0^{\sqrt{\frac{\pi}{2}}} x \cos(x^2)\, e^{\,\sin(x^2)} dx.$$

[2]

3

Find the inverse of the matrix

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}.$$

[2]

4

Find the real and imaginary parts of the complex number

$$\frac{2+3i}{4-5i}.$$

[2]

5

Give the first three terms of the Taylor series expansion of the function $f(x) = \ln{(x^2)}$ about the point $x = 2$. [2]

6

Find the value of ω for which $f(x,t) = \exp(-2x + \omega t)$ is a solution of the differential equation

$$\frac{\partial^2 f}{\partial x^2} = 6\,\frac{\partial f}{\partial t}.$$

[2]

7

The probabilities that events A and B occur are $\frac{1}{3}$ and $\frac{1}{2}$, respectively. Under the assumption that if A occurs, then B definitely occurs, find the conditional probability $P(A|B)$. [2]

8

Calculate the divergence and the curl of the vector field

$$\boldsymbol{F}(x,y,z) = \left(x^2 - y^2\right)\hat{\boldsymbol{\imath}} + xy\,\hat{\boldsymbol{\jmath}} + \left(3z^2 - x^2\right)\hat{\boldsymbol{k}}.$$

[2]

9

Evaluate the line integral of the vector field $\boldsymbol{G}(x,y,z) = xy^2\,\hat{\boldsymbol{\imath}} + x^2y\,\hat{\boldsymbol{\jmath}}$ along the straight line joining the points $(0,0,0)$ and $(1,1,0)$. [2]

10

Find the stationary points of the function $f(x) = \sin(x^2 - 2)$ in the interval $-2 < x < 2$. [2]

SECTION B

11X

Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

(a) Show that $\lambda_1 = 4$ is an eigenvalue of $\boldsymbol{A}$ and find the corresponding eigenvector $\boldsymbol{e}_1$. [4]

(b) Show that any vector $\boldsymbol{e}_2$ that is perpendicular to $\boldsymbol{e}_1$ is also an eigenvector of $\boldsymbol{A}$ and find the corresponding eigenvalue. [3]

(c) Find an orthonormal set of eigenvectors of $\boldsymbol{A}$. [2]

(d) Relate the eigenvalues to the trace and determinant of $\boldsymbol{A}$. [2]

(e) Write down $\boldsymbol{B}$, the matrix whose columns are the eigenvectors of $\boldsymbol{A}$. [2]

(f) Calculate $\boldsymbol{B}^{-1}$, the inverse of $\boldsymbol{B}$. [4]

(g) Calculate $\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}$ and comment on the result. [3]

12W

Let

$$I_{n,k}(x,y) = \int_y^x \cos^k u \left(1 - \sin^3 u\right)^n du,$$

where n and k are non-negative integers, and x and y are real variables.

(a) Without evaluation of the integral, calculate $\frac{\partial I_{n,k}(x,y)}{\partial x}$, $\frac{\partial I_{n,k}(x,y)}{\partial y}$, and $\frac{\partial^2 I_{n,k}(x,y)}{\partial x \partial y}$. [3]

(b) For $k = 1$, using a suitable substitution and then integrating by parts, or otherwise, derive the reduction formula connecting $I_{n,1}(\pi/2, 0)$ and $I_{n-1,1}(\pi/2, 0)$, that is, find $f(n)$ in the relation

$$I_{n,1}\left(\frac{\pi}{2}, 0\right) = f(n)\, I_{n-1,1}\left(\frac{\pi}{2}, 0\right),$$

for $n > 0$. [10]

(c) Calculate $I_{3,1}(\pi/2, 0)$. [2]

(d) Find $I_{n,1}(\pi/2, 0) - I_{n,3}(\pi/2, 0)$ in terms of n. [5]

13Z

(a) Find the solution of

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0,$$

subject to $dy/dx = -1$ at $x = 0$ and $y = 1$ at $x = 0$. [6]

(b) Find the general solution of

$$\frac{d^2y}{dx^2} + 6\,\frac{dy}{dx} + 9y = 1 + 3x.$$

[7]

(c) Find the general solution of

$$\frac{d^2y}{dx^2} + 3\,\frac{dy}{dx} + 2y = x + e^{3x}.$$

[7]

14S

In three dimensions, points O, P, Q and R have Cartesian coordinates $(0,0,0)$, $(2,1,0)$, $(1,2,0)$, $(0,1,1)$, respectively.

(a) A tetrahedron has vertices O, P, Q and R.

(i) Find the cosine of the angle between the faces OPR and ORQ of the tetrahedron. [5]

(ii) Find the vector area of the face PQR of the tetrahdron in the form $A\hat{\boldsymbol{n}}$ where $\hat{\boldsymbol{n}}$ is the outward pointing unit normal vector. Hence calculate the sum of the vector areas of the other three faces. [6]

(iii) Find the shortest distance from the origin to the plane containing PQR. [5]

(b) Find the volume of the parallelepiped formed from the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$. [4]

15T

Let an electric field $\boldsymbol{E}$ in three dimensions be given by

$$\boldsymbol{E} = \begin{cases} -\dfrac{k}{a^3}\,\boldsymbol{r} & \text{for } r < a \\[2ex] -\dfrac{k}{r^3}\,\boldsymbol{r} & \text{for } r \geqslant a \end{cases}$$

where k and a are positive constants, and $r = |\boldsymbol{r}|$.

(a) Show that $\boldsymbol{E}$ is conservative inside and outside the sphere of radius a centred at the origin. [8]

(b) The electric potential $V(r)$ is defined by

$$\boldsymbol{E} = -\frac{dV}{dr}\,\frac{\boldsymbol{r}}{r}.$$

Derive the general expressions for $V(r)$ inside and outside the sphere of radius a centred at the origin. [7]

(c) By requiring V to be continuous at all points and to vanish at infinity, find the constants of integration that appear in the general expressions for the potentials. [5]

16Y

(a) Consider the function

$$F(x,y,z) = x^3yz + xy + z + 3,$$

where $x = 3\cos t$, $y = 3\sin t$, and $z = 2t$. Differentiate F with respect to t and evaluate dF/dt at $t = \pi/2$. [6]

(b) What value of a makes the differential form

$$\left(x^2 + xy - y^2\right) dx + \left(\frac{1}{2}\,x^2 - axy\right) dy$$

exact? [7]

(c) Using the value of a determined in part (b), find the solution of the differential equation

$$\left(x^2 + xy - y^2\right) dx + \left(\frac{1}{2}\,x^2 - axy\right) dy = 0.$$

[7]

17S

(a) State Taylor's theorem for the expansion of a differentiable function $f(x)$ about the point $x = a$. Write down explicit expressions for the first four terms and the remainder term. [4]

(b) Taking $f(x) = x^{1/2}$, find an approximation for $2^{1/2}$ as a sum of fractions using the first four terms of the Taylor expansion of $f(x)$ about $x = 1$. By considering the expression for the remainder term in this case, show that the absolute value of the error is not larger than $\frac{5}{128}$. [10]

(c) Find the expansion of $\ln(1 + e^x)$ about $x = 0$ up to and including terms in x^3. [6]

18R

On Monday, Jack puts 10 pears (8 green and 2 red) and 8 apples (5 green and 3 red) in a basket after which Sarah picks one fruit at random from the basket and does not put it back. Denote by A the event of picking out a green fruit and by B the event of picking out a pear.

(a) Find $P(A)$, $P(B)$, $P(A|B)$, $P(B|A)$ and $P(A \cap B)$. [8]

(b) Using your results from part (a), verify that Bayes' theorem is satisfied. [2]

(c) On Tuesday Sarah picks one more fruit. Find the probability of it being a red apple. [5]

(d) Given that Sarah picked a red apple on Tuesday, find the probability that she picked a green pear on Monday. [5]

19Z*

(a) Write down the derivative of the integral

$$I(a) = \int_0^a f(x,a)\,dx$$

with respect to a.

Determine the derivative dI/da for the case $f(x,a) = e^{-ax}$ and check your answer by evaluating $I(a)$ directly, and then differentiating your result with respect to a. [8]

(b) Differentiate the integral

$$J(b) = \int_0^1 \frac{x^b - 1}{\ln x}\,dx, \quad (b > -1)$$

with respect to the parameter b, and hence show that

$$J(b) = \ln(b+1).$$

[12]

20T*

Consider the heat flow equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

which governs the temperature $u(x,t)$ of a thin rod.

(a) Verify that

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\varphi(\xi)}{\sqrt{t}} \; \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi, \tag{2}$$

where φ is an arbitrary function, is a solution of (1).

[You may assume that φ and all its derivatives exist for all values of ξ and that the integral on the RHS of (2) converges.] [7]

(b) By making the substitution

$$\eta = \frac{\xi - x}{2\sqrt{t}}$$

on the RHS of (2), or otherwise, show that

$$\lim_{t \to 0} u(x,t) = \varphi(x).$$

[7]

(c) Find the solution $u(x,t)$ of the heat flow equation for a given initial temperature distribution $\varphi(x) = ax$. [6]

END OF PAPER