UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday, 3 June, 2011 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the left hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

Consider the eigenvalue problem

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda r(x)y = 0\,, \quad 0 \leqslant x \leqslant \infty\,, \qquad (*)$$

where $p(x) \to 0$ as $x \to 0$ and as $x \to \infty$, and $r(x) > 0$ such that $r(x) \to 0$ as $x \to \infty$.

Show that eigenfunctions y_m and y_n, associated respectively with distinct eigenvalues λ_m and λ_n, satisfy the orthogonality property

$$\int_0^\infty r(x)\, y_m\, y_n\, dx = 0\,, \qquad (\dagger)$$

making it clear where you have to make assumptions about the behaviour of y_m and y_n as $x \to 0$ and $x \to \infty$. [6]

Show that the equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0 \qquad (\ddagger)$$

may be written in the form $(*)$ and find the corresponding functions $p(x)$ and $r(x)$. Write down the orthogonality property satisfied by y_m and y_n in this case. [6]

You are given that $(\ddagger)$ has eigenvalues $\lambda_0,\ \lambda_1,\ \lambda_2,\ \ldots,$ where $\lambda_n = n$, and corresponding eigenfunctions $y_0,\ y_1,\ y_2,\ \ldots$ where each y_n is a polynomial of degree n.

Find the functions y_1 and y_2, respectively linear and quadratic polynomials, and show explicitly that they satisfy the orthogonality property $(\dagger)$. [8]

2C

The temperature $T(\mathbf{x})$ in a volume $\mathcal{V}$ of a solid satisfies the steady-state diffusion equation

$$\nabla^2 T = 0\,, \qquad (*)$$

in $\mathcal{V}$. T is specified to be a given function $T_b(\mathbf{x})$ on the boundary of $\mathcal{V}$, the surface $\mathcal{S}$. Show that the solution of $(*)$ with the boundary condition $T = T_b(\mathbf{x})$ on $\mathcal{S}$ is unique. [7]

A solid body consists of one substance in the volume $\mathcal{V}_1$ entirely enclosed within a second substance occupying the volume $\mathcal{V}_2$. The outer surface of $\mathcal{V}_2$ is $\mathcal{S}_2$. The outer surface of $\mathcal{V}_1$ and also the inner surface of $\mathcal{V}_2$ is $\mathcal{S}_1$. In this case the temperature T satisfies the equations

$$\nabla^2 T = 0 \;\text{ in } \mathcal{V}_1\,,$$
$$\nabla^2 T = 0 \;\text{ in } \mathcal{V}_2\,,$$

with T a given function $T_b(\mathbf{x})$ on $\mathcal{S}_2$, T continuous across $\mathcal{S}_1$ and $\alpha \boldsymbol{\nabla} T.\mathbf{n}|_{\mathcal{S}_1^+} = \beta \boldsymbol{\nabla} T.\mathbf{n}|_{\mathcal{S}_1^-}$ where the unit vector $\mathbf{n}$ is the outward normal to $\mathcal{S}_1$, $|_{\mathcal{S}_1^+}$ denotes the limit as $\mathcal{S}_1$ is approached from $\mathcal{V}_2$ and $|_{\mathcal{S}_1^-}$ denotes the limit as $\mathcal{S}_1$ is approached from $\mathcal{V}_1$. α and β are positive constants.

Show that the above equations and boundary conditions have a unique solution in $\mathcal{V}_1$ and $\mathcal{V}_2$. [13]

[*Hint: Start by applying the approach you used in the first part of the question to* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *separately.*]

3C

Derive the fundamental solution

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log |\mathbf{x} - \mathbf{y}| \,,$$

to the Poisson equation

$$-\nabla^2 G = \delta(\mathbf{x} - \mathbf{y}) \,,$$

in two-dimensional space. [4]

A point charge q is placed at the origin in the presence of a boundary along the line $x = b$, with $b > 0$. The electrostatic potential $V(\mathbf{x})$ satisfies

$$\nabla^2 V = q\delta(\mathbf{x}) \,.$$

Using the method of images, construct an expression for $V(\mathbf{x})$ in $x < b$ for the cases:

i. where the boundary is an earthed conductor, i.e. $V = 0$ on the boundary;

ii. where the boundary is an insulator, i.e. $\mathbf{n}.\boldsymbol{\nabla} V = \partial V/\partial x = 0$ on the boundary. ($\mathbf{n}$ is the unit normal to the boundary.)

In each case show explicitly that your solution satisfies the required boundary conditions. [8]

Now consider the case where there are two insulating boundaries at $x = b$ and at $x = -b$. Construct the corresponding image system which gives an appropriate solution for V in $-b < x < b$. Write down expressions for V and for $\partial V/\partial y$ as series. [6]

Deduce that for $y > 0$,

$$\int_{-b}^{b} \frac{\partial V}{\partial y}\, dx = \frac{q}{2\pi} \sum_{n=-\infty}^{\infty} \left\{ \tan^{-1}\left[\frac{(2n+1)b}{y}\right] - \tan^{-1}\left[\frac{(2n-1)b}{y}\right] \right\} = \frac{q}{2} \,.$$

[2]

4C

State the residue theorem for the integral

$$\oint_{\mathcal{C}} f(z)\, dz\,,$$

where $\mathcal{C}$ is a closed contour and $f(z)$ is analytic within $\mathcal{C}$ except for a finite number of poles at $z_1,\, z_2,\, \ldots,\, z_N$. [4]

Consider the function $f(z) = \log z/(1 + z^{\beta})$ where $\beta > 1$. Show that a branch cut can be chosen so that, apart from isolated poles, $f(z)$ is analytic in the region $|z| > 0$, $0 \leqslant \arg(z) < \pi$. Identify all poles of $f(z)$ in this region and calculate the corresponding residues. [8]

Now, by considering the integral of $f(z)$ around a suitable closed contour which includes the real axis from $z = \epsilon$ ($\epsilon \ll 1$) to $z = R$ ($R \gg 1$) and which encloses a single pole of $f(z)$, show that

$$\int_0^{\infty} \frac{\log x}{1 + x^{\beta}}\, dx = -\frac{\pi^2}{\beta^2} \frac{\cos(\pi/\beta)}{\sin^2(\pi/\beta)}\,,$$

and

$$\int_0^{\infty} \frac{1}{1 + x^{\beta}}\, dx = \frac{\pi}{\beta \sin(\pi/\beta)}\,.$$

[8]

5C

The Fourier transform $\tilde{f}(k)$ of a function $f(t)$ and the corresponding inverse transform are defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikt} f(t)\, dt\,, \;\; f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikt} \tilde{f}(k)\, dk\,.$$

Show that if

$$h(t) = \int_{-\infty}^{\infty} f(s) g(t-s)\, ds,$$

then $\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$. [6]

Consider the differential equation

$$\frac{d^2y}{dt^2} + (a+b)\frac{dy}{dt} + aby = f(t)\,,$$

with the constants a and b such that $a > b > 0$. The solution $y(t)$ and its derivatives may be assumed to tend to zero as $t \to \pm\infty$.

Derive an equation relating $\tilde{y}$ to $\tilde{f}$, carefully justifying all steps in your calculation. Deduce that

$$y(t) = \int_{-\infty}^{\infty} f(s) G(t-s)\, ds\,,$$

and find the function $G(t)$ by inverting $\tilde{G}(k)$. [7]

In the case $f(t) = e^{-|t|}$ evaluate $\tilde{f}(k)$. Assuming that $a \neq 1$ and $b \neq 1$, deduce an expression for $\tilde{y}(k)$ and invert to deduce $y(t)$. [7]

6C

[*In this question you should assume three dimensions.*]

Write down the transformation law for the components t_{ij} of a tensor of rank 2 and the components u_{ijkl} of a tensor of rank 4 under rotation of the coordinate axes. [2]

What is an *isotropic tensor*? Show that $a\delta_{ij}$ and $b\delta_{ij}\delta_{kl} + c\delta_{ik}\delta_{jl} + d\delta_{il}\delta_{jk}$, with a, b, c and d constants and δ_{ij} the Kronecker delta, are isotropic tensors. [4]

By considering rotations by $\pi/2$ about two different coordinate axes show that $a\delta_{ij}$ is the most general form of an isotropic tensor of rank 2. [4]

(You may henceforth assume that $b\delta_{ij}\delta_{kl} + c\delta_{ik}\delta_{jl} + d\delta_{il}\delta_{jk}$ is the most general form for an isotropic tensor of rank 4.)

Consider the tensors

$$A_{ij} = \int_{\mathcal{V}_R} x_i x_j |\mathbf{x}|^2 \, dV \, ,$$

and

$$B_{ijkl} = \int_{\mathcal{V}_R} x_i x_j x_k x_l \, dV \, ,$$

where the volume $\mathcal{V}_R$ is a sphere of radius R centred on the origin. Show that A_{ij} and B_{ijkl} are isotropic tensors and determine their components. [6]

Consider the tensor

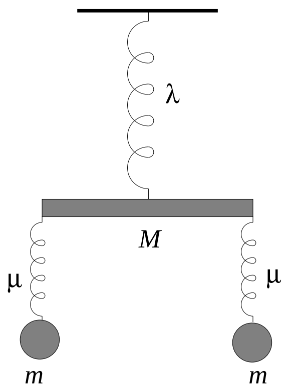
$$C_{ij} = \int_{\mathcal{V}_R} \{x_i x_j |\mathbf{x}|^2 + x_i x_j (\mathbf{x}.\mathbf{n})^2\} \, dV \, ,$$

where $\mathbf{n}$ is a fixed unit vector. What are the eigenvectors and corresponding eigenvalues of this tensor? [4]

7C

A mechanical system has three degrees of freedom and is described by coordinates q_1, q_2 and q_3, where $q_1 = q_2 = q_3 = 0$ corresponds to a position of equilibrium of the system. The kinetic energy $\mathcal{T} = \frac{1}{2}\sum_{i=1}^{i=3}\sum_{j=1}^{j=3} T_{ij}(\mathbf{q})\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{T}\dot{\mathbf{q}}$, defining a matrix $\mathbf{T}$. The potential energy is given by the function $\mathcal{V}(\mathbf{q})$.

Define the Lagrangian $\mathcal{L}$ of the system and write down the corresponding Euler-Lagrange equations. What conditions must apply at the equilibrium position $q_1 = q_2 = q_3 = 0$? Calculate the leading-order non-constant terms in a Taylor expansion of $\mathcal{V}(\mathbf{q})$ about this position, and hence show that these leading-order non-constant terms can be written as $\frac{1}{2}\mathbf{q}^T\mathbf{V}\mathbf{q}$ for some constant matrix $\mathbf{V}$. Deduce the form of the Lagrangian and the corresponding Euler-Lagrange equations for small disturbances from equilibrium. With reference to this set of equations define the terms *normal frequencies* and *normal modes*. [5]



Consider a system consisting of a heavy horizontal bar of mass M, from the ends of which two masses m hang on identical vertical springs, each with spring constant μ and equilibrium length l. The bar itself is suspended from a fixed point by a spring with spring constant λ and equilibrium length L. (The bar is constrained to remain horizontal.) Define q_1, q_2 and q_3 to be the vertical displacements of, respectively, the bar and the two masses away from their equilibrium positions. Show that the relevant matrices $\mathbf{T}$ and $\mathbf{V}$ (as defined above) take the form

$$\mathbf{T} = \begin{pmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \frac{\lambda}{L} + \frac{2\mu}{l} & -\frac{\mu}{l} & -\frac{\mu}{l} \\ -\frac{\mu}{l} & \frac{\mu}{l} & 0 \\ -\frac{\mu}{l} & 0 & \frac{\mu}{l} \end{pmatrix},$$

and hence construct the Lagrangian for this system. [6]

Hence for the case $M = 2$, $m = 1$, $\lambda = 4$, $\mu = 1$, $L = 1$ and $l = 1$, derive the corresponding normal frequencies and normal modes. [6]

Give a brief geometrical description of each normal mode. [3]

8B

Define the *order* $|G|$ of a finite group G and the *order* of an element $g \in G$. [2]

Consider the Cartesian product of two groups G_1 and G_2. This is the set $G_1 \times G_2$ of all pairs (g_1, g_2) with the composition law

$$(g_1, g_2)(g_1', g_2') \equiv (g_1 g_1', g_2 g_2').$$

Show that $G_1 \times G_2$ is a group. What is the order of the group? [6]

Consider the order 2 group $Z_2 = \{e, w\}$. Construct the multiplication table for the order 4 group $Z_2 \times Z_2$. [4]

Now consider the order 4 cyclic group $Z_4 = \{e, a, a^2, a^3\}$. Show that the order 2 cyclic group $Z_2 = \{e, a^2\}$ is a proper subgroup for Z_4. Prove there are no other proper subgroups for Z_4. Hence, show that $Z_2 \times Z_2$ is not isomorphic to Z_4. [8]

9B

If H and K are subgroups of G, show that the intersection $H \cap K$ is also a group. [6]

Consider D_3, the group of symmetries of the equilateral triangle. D_3 has six elements: the identity (I); two rotations (A, B); and three reflections (C, D, E). Explain their geometrical action on the equilateral triangle. Construct the multiplication table for D_3. Is this group abelian? [9]

How many order 2 subgroups are there in D_3? List them. Are they normal subgroups of D_3? Justify your conclusion. Finally, show by explicit construction that the union of two order 2 subgroups of D_3 does **not** form a group. [5]

10B

[*In this question, you may state without proof any theorems you use.*]

Suppose G_1 and G_2 are groups, and D is a mapping $D : G_1 \to G_2$. Give the definition for the map D to be a *homomorphism*. What is the *kernel* of D, $\mathrm{Ker}(D)$? [2]

Let $S_3 = \{e, x, y, y^2, xy, xy^2\}$ be the $n = 3$ symmetric group with

$$x^2 = y^3 = e\,,\ yx = xy^2\,,\ y^2x = xy\,.$$

Verify by explicit calculation that, with $z = \exp(2\pi i/3)$,

$$\begin{aligned} R(x) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R(y) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix}, \quad R(y^2) = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix}, \qquad (*) \\ R(xy) &= \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix}, \quad R(xy^2) = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix}, \end{aligned}$$

is a two-dimensional complex representation of S_3. You can assume without proof that $(*)$ is an irreducible representation. [7]

Given that the trivial representation of S_3 is the one-dimensional complex representation $T(s) = 1$ where $s \in S_3$, find the non-trivial one-dimensional complex representation $U : S_3 \to \mathbb{C}^1$. What is $\mathrm{Ker}(U)$? Is U a *faithful* representation? [8]

Finally, given these results, deduce the number of conjugacy classes in S_3. [3]

END OF PAPER