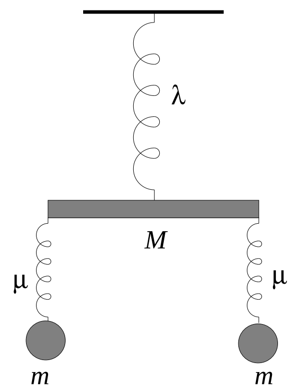
7C

A mechanical system has three degrees of freedom and is described by coordinates q_1, q_2 and q_3, where $q_1 = q_2 = q_3 = 0$ corresponds to a position of equilibrium of the system. The kinetic energy $\mathcal{T} = \frac{1}{2}\sum_{i=1}^{i=3}\sum_{j=1}^{j=3} T_{ij}(\mathbf{q})\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{T}\dot{\mathbf{q}}$, defining a matrix $\mathbf{T}$. The potential energy is given by the function $\mathcal{V}(\mathbf{q})$.

Define the Lagrangian $\mathcal{L}$ of the system and write down the corresponding Euler-Lagrange equations. What conditions must apply at the equilibrium position $q_1 = q_2 = q_3 = 0$? Calculate the leading-order non-constant terms in a Taylor expansion of $\mathcal{V}(\mathbf{q})$ about this position, and hence show that these leading-order non-constant terms can be written as $\frac{1}{2}\mathbf{q}^T\mathbf{V}\mathbf{q}$ for some constant matrix $\mathbf{V}$. Deduce the form of the Lagrangian and the corresponding Euler-Lagrange equations for small disturbances from equilibrium. With reference to this set of equations define the terms *normal frequencies* and *normal modes.* [5]



Consider a system consisting of a heavy horizontal bar of mass M, from the ends of which two masses m hang on identical vertical springs, each with spring constant μ. The bar itself is suspended from a fixed point by a spring with spring constant λ. (The bar is constrained to remain horizontal.) Define q_1, q_2 and q_3 to be the vertical displacements of, respectively, the bar and the two masses away from their equilibrium positions. Show that the relevant matrices $\mathbf{T}$ and $\mathbf{V}$ (as defined above) take the form

$$\mathbf{T} = \begin{pmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \lambda + 2\mu & -\mu & -\mu \\ -\mu & \mu & 0 \\ -\mu & 0 & \mu \end{pmatrix},$$

and hence construct the Lagrangian for this system. [6]

Hence for the case $M = 2$, $m = 1$, $\lambda = 4$ and $\mu = 1$, derive the corresponding normal frequencies and normal modes. [6]

Give a brief geometrical description of each normal mode. [3]