UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday, 26 May, 2009 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the left hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover shett
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C a) Define the *divergence* and *curl* of a vector field $\mathbf{F}(\mathbf{x})$ in terms of its components F_i, $i = 1, 2, 3$ in Cartesian coordinates $\mathbf{x} = (x, y, z)$. For two vector fields $\mathbf{F}(\mathbf{x})$ and $\mathbf{G}(\mathbf{x})$
[6] prove that

$$\nabla \times (\mathbf{F} \times \mathbf{G}) \;=\; (\mathbf{G} \cdot \nabla)\,\mathbf{F} \;-\; \mathbf{G}\,(\nabla \cdot \mathbf{F}) \;-\; (\mathbf{F} \cdot \nabla)\,\mathbf{G} \;+\; \mathbf{F}\,(\nabla \cdot \mathbf{G})\,.$$

b) The divergence theorem states that, for any vector field $\mathbf{F}(\mathbf{x})$

$$\int_V (\nabla \cdot \mathbf{F})\; dV \;=\; \int_S \mathbf{F} \cdot d\mathbf{S},$$

where V is a volume in three dimensional space and S the surface bounding it. Evaluate the integrals on both sides explicitly in the case

$$\mathbf{F}(\mathbf{x}) \;=\; (xy^2, yz^2, zx^2),$$

where V is a cube with sides of unit length parallel to the coordinate axes and center at
[14] the origin.

2A Let $u(x, t)$ denote the displacement of a string which is stretched between the points $x = 0$ and $x = \pi$ and fixed at these points. Suppose that the string is subject to a resistance which is proportional to the velocity at each point. The string starts from rest in the position $u(x, 0) = f(x)$ and the time variable t is scaled so that u satisfies the following equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - 2h\frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0,$$

where $h < 1$ denotes the resistance coefficient.

Use separation of variables to express $u(x, t)$ in terms of an appropriate infinite
[12] series.

[8] Compute explicitly the coefficients of the above series in the case that

$$f(x) = \begin{cases} x, & 0 < x < \pi/2, \\ -x + \pi, & \pi/2 < x < \pi. \end{cases}$$

3A Find an integral representation of the solution of the following ODE by first
[12] computing the associated Green's function:

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - u = f(x), \quad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

[8] Use the expression obtained to find an explicit solution in the case that

$$f(x) = x, \quad 0 < x < 1.$$

4A Let $f_1(x)$ and $f_2(x)$ be given functions. Derive the Fourier transform of the convolution integral

$$\int_{-\infty}^{\infty} f_1(y)f_2(x-y)dy, \quad x \in \mathbb{R},$$

[8] in terms of the Fourier transforms of the functions $f_1(x)$ and $f_2(y)$.

[12] Use the above result to find the inverse Fourier transform of the function

$$F(k) = \frac{1}{(k^2+1)^2}, \quad k \in \mathbb{R}.$$

5B MathMethodsIII

Explain how a real symmetric $n \times n$ matrix a_{ij} can be diagonalized by an orthogonal matrix O_{ij}. [5]

Show that the quadratic form

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j \tag{1}$$

can be put into the form

$$\sum_{i=1}^{n} \lambda_i y_i^2$$

by an orthogonal transformation $y_i = \sum_{j=1}^{n} O_{ij} x_j$. [5]

Deduce further that the quadaratic form (**??**) can be put into the form

$$z_1^2 + z_2^2 + \cdots + z_m^2 - z_{m+1}^2 - \cdots - z_l^2 \tag{2}$$

by a real linear transformation $z_i = \sum_{ij} L_{ij} x_j$ for some positive integers m, l such that $1 \leqslant m \leqslant l \leqslant n$. [5]

By computing L_{ij} explicitly, put the quadratic form

$$x_1 x_2 + x_1 x_3$$

into the form (**??**). [5]

6B Define a unitary matrix. Show that if μ is an eigenvalue of a unitary matrix then $|\mu| = 1$. [2]

Let A be an $n \times n$ complex Hermitian matrix, with a set of n linearly independent eigenvectors $\mathbf{e}_j$:

$$A\mathbf{e}_j = \lambda_j \mathbf{e}_j.$$

Prove that the eigenvalues λ_j are real. [2]

Let I denote the identity matrix. Show that $(A + iI)$ is invertible, and that

$$V = (A - iI)(A + iI)^{-1}$$

is unitary. What are the eigenvalues of V? [8]

Let U be unitary, and assume that 1 is not an eigenvalue, so that $I - U$ is invertible. Show that the matrix

$$B = i(I + U)(I - U)^{-1}$$

is Hermitian. Hence prove that U is diagonalizable. [8]

[You may assume without proof that Hermitian matrices are diagonalizable.]

7A

(a) Using the fact that

$$\frac{1}{12} = \frac{1}{3} - \frac{1}{4},$$

[4] compute

$$2\sqrt{2}\exp\left(\frac{i\pi}{12}\right).$$

(b) Find the first three terms of the Taylor series expansion at $z = 0$ of the following
[5] function:

$$f(z) = \frac{1}{1 + z + z^2}, \quad z \in \mathbb{C}.$$

[5] (c) Find the residue at $z = \pi$ of the following function:

$$f(z) = \left[\cosh\left(\frac{1}{z - \pi}\right)\right]^2, \quad z \in \mathbb{C}.$$

[6] (d) Compute all possible values of $|z^z|$ by using the polar representation of z.

8A

(a) The ODE

$$\frac{du}{dx} + f(x)u = g(x)u^{\alpha}, \quad \alpha \neq 1,$$

where α is a real number, is called Bernoulli's equation. Use the transformation

$$v(x) = (u(x))^{1-\alpha}$$

to map the above equation into a linear ODE for $v(x)$. Then use this result to find the
[10] general solution of the following equation:

$$\frac{du}{dx} + xu = xu^3.$$

(b) Find the first three terms of the series solution of the ODE

$$z\frac{d^2w}{dz^2} + \frac{dw}{dz} + 4zw = 0,$$

[10] which is analytic at $z = 0$.

9C The function $y(x)$ is a stationary point of

$$F = \int_a^b \left[p(x) y'(x)^2 + q(x) y(x)^2 \right] dx$$

subject to the condition $G = 1$, where

$$G = \int_a^b w(x)\, y(x)^2 \, dx$$

and with boundary conditions $y(a) = y(b) = 0$. Here $p(x)$, $q(x)$ and $w(x)$ are real functions defined on the interval $a \leqslant x \leqslant b$ and $w(x) > 0$.

Show that $y(x)$ satisfies the Sturm-Liouville equation

$$-\frac{d}{dx}\left[p(x) \frac{dy}{dx} \right] + q(x) y = \lambda w(x) y$$

[10] with eigenvalue $\lambda = F$.

The function $\psi(r)$ satisfies the equation

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} + E\psi = 0$$

in the interval $0 \leqslant r \leqslant 1$ with boundary conditions $\psi(1) = 0$ and $\psi(0)$ finite. Rewrite this as a Sturm-Liouville equation for $\sigma(r) = r\psi(r)$ and use the Rayleigh-Ritz method with trial solutions corresponding to

$$\psi^{(1)}(r) = (1 - r)$$
$$\psi^{(2)}(r) = (1 - r^2)$$

to obtain two estimates E_1 and E_2 of the lowest value $E_{\min}$ of E for which a solution exists.

Which of the two estimates is more accurate and why?

Explain how you would proceed to find an estimate of $E_{\min}$ which is more accurate than

[10] both E_1 and E_2.

10A

(a) Let the given functions f and g be twice differentiable with respect to their arguments. Let $y = y(x)$ be a continuously differentiable function such that $y(x_1) = y_1$ and $y(x_2) = y_2$, where x_1, x_2, y_1, y_2 are given. Derive the differential equation which must be satisfied by the function $y(x)$ which renders the integral I an extremum, while the integral J possesses a given prescribed value, where

$$I = \int_{x_1}^{x_2} f\left(x, y, \frac{dy}{dx}\right) dx, \quad J = \int_{x_1}^{x_2} g\left(x, y, \frac{dy}{dx}\right) dx.$$

[You may use the fact that the requirement that the function $F(\epsilon_1, \epsilon_2)$ is an extremum under the constraint that the function $G(\epsilon_1, \epsilon_2)$ is constant, is equivalent to the requirement that the function $F + \lambda G$ is an extremum, where the constant λ is called Lagrange's
[10] multiplier.]

(b) By generalizing the result of (a) it can be shown that if I and J are given by

$$I = \int_{t_1}^{t_2} f(x(t), y(t), \dot{x}(t), \dot{y}(t), t)dt, \quad J = \int_{t_1}^{t_2} g(x(t), y(t), \dot{x}(t), \dot{y}(t), t)dt,$$

then the relevant differential equations are the following:

$$\frac{\partial \tilde{f}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \tilde{f}}{\partial \dot{x}}\right) = 0, \quad \frac{\partial \tilde{f}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \tilde{f}}{\partial \dot{y}}\right) = 0,$$

where $\tilde{f} = f + \lambda g$.

Use this result to obtain the differential equations describing the classical isoperimetric problem, namely the problem of determining the closed non-self-intersecting plane curves of given length which enclose the greatest possible area. This corresponds to the following choices for f and g:

$$f = \frac{1}{2}(x\dot{y} - y\dot{x}), \quad g = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

[10] Solve the resulting equations to show that the relevant curves are circles.

END OF PAPER