UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 10 June, 2009 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11S****). Answers to these questions should be tied up in* **separate** *bundles, marked* ***R, S, T, X, Y*** *or* ***Z*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* **must** *be completed and attached to each bundle, with the appropriate letter* ***R, S, T, X, Y*** *or* ***Z*** *written in the section box.*

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
Approved calculators allowed.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Show that if **a** and **b** are two non-parallel non-zero vectors then

$$x\mathbf{a} + y\mathbf{b} = \mathbf{0}$$

implies $x = y = 0$. [2]

2 Calculate:

(i)

$$\int_{-\pi}^{\pi} x^2 \sin x \, dx;$$

[1]

(ii)

$$\int_{0}^{\pi} (\cos^2 x - \sin^2 x) dx.$$

[1]

3 Given the two functions

$$f = x + y + z$$
$$g = x^2 + y^2 + z^2$$

find the point (x, y, z) for which $\boldsymbol{\nabla} f = \boldsymbol{\nabla} g$. [2]

4 Given the matrix

$$\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$$

find all α such that one eigenvalue is 0. [1]

5 Find the zeros, the stationary points and the inflection points of

$$y = x^3 - 3x^2 + 4$$

and state whether the stationary points are maxima or minima. Indicate these points on a graph of the function. [3]

6 Show that

$$u(x,t) = (x-ct)^2 + (x+ct)^2$$

is a solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where c is a constant. [2]

7 Determine the equation of the line that is tangent to the curve $y = \ln x$ at $x = 1$. [2]

8 Given that, for small x, $e^x \simeq 1 + x + \frac{1}{2}x^2$ and $\ln(1+x) \simeq x - \frac{1}{2}x^2$ find an expression for $\ln(1+e^x)$ ignoring powers of x greater than 2. [2]

9 Find the values of θ in $[0, 2\pi]$ for which

$$|\sqrt{2}\cos\theta| > 1.$$

[2]

10 Show that the probability distribution

$$f(k,\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

for non-negative integer k is normalised, i.e.

$$\sum_{k=0}^{\infty} f(k,\lambda) = 1.$$

[2]

SECTION B

11S

(a) Write the equation for the straight line joining the points $\mathbf{a} = (1, 2, 4)$ and $\mathbf{b} = (2, 4, 2)$ in both vector and Cartesian form.

Find the position vector of the point where this line intersects the (x, y)-plane. [5]

(b) Find the vector equation for the line of intersection of the two planes

$$\begin{aligned} 2x - y - z &= 3 \\ 3x - y - 3z &= 4. \end{aligned}$$

[8]

(c) Find the shortest distance from the position $(2, 2, 1)$ to the line of intersection of the planes found in (b). [7]

12T The position $y(t)$ of a particle of unit mass suspended at the end of a spring of constant k moving under the influence of a friction force proportional to the speed satisfies

$$\frac{d^2y}{dt^2} = -ky - \lambda \frac{dy}{dt},$$

with k and λ positive constants.

(i) Find the general solution of this equation and state the condition under which the motion is oscillatory. [6]

(ii) Assuming that this condition is satisfied, state the angular frequency of the oscillation and find the time for the amplitude of the oscillation to decrease by a quarter. [4]

(iii) Suppose now that $\lambda = 0$ and $k = \omega^2 > 0$, that $y = dy/dt = 0$ at $t = 0$, and that an external force $F(t) = 2te^{-t}$ acts on the particle for $t > 0$. Find the amplitude of the resulting oscillation as $t \to \infty$ by solving

$$\frac{d^2y}{dt^2} + \omega^2 y = 2te^{-t}$$

subject to the initial conditions. [10]

13X The function $z(x,y)$ is given by

$$z(x,y) = (x-1)y \exp\left[-\tfrac{1}{2}\left((x-1)^2 + y^2\right)\right] .$$

(a) Do the following in whichever order you find most convenient, and using any (clearly stated) methods of your choice:

(i) Find the positions of the stationary points of $z(x,y)$.

(ii) Classify each stationary point as either a maximum, a minimum, or a saddle point.

(iii) Sketch contours of z in the (x,y)-plane, indicating the locations of the stationary points. [13]

(b) Suppose that $z(x,y)$ describes a surface in a three-dimensional space in which x, y and z are the usual Cartesian co-ordinates.

(i) Determine the unit vector which is normal to the surface at the point where $x = y = 1$. [4]

(ii) An ant is discovered on the surface at the point where $x = y = 1$, and is found to be walking directly downhill (by the steepest possible path) with speed u. Determine its velocity vector $\mathbf{v} = (v_x, v_y, v_z)$ at that moment. [2]

(iii) If the ant continues to walk directly downhill for as long as possible, where will it end up? [1]

14Y

(a) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$$

(i) Calculate $\det A$. [2]

(ii) Calculate A^{-1}. [4]

(b) Find the value of λ for which the following set of linear equations has non-zero solutions

$$\begin{aligned} x + y + z &= 0 \\ x + 2y &= 0 \\ x - 3y + \lambda z &= 0. \end{aligned}$$

[7]

(c) Find all conditions on the constants a, b and c that allow the set of equations

$$\begin{aligned} x + y + z &= 0 \\ ax + by + cz &= 0 \\ a^2x + b^2y + c^2z &= 0 \end{aligned}$$

to have non-zero solutions. [7]

15R

(a) Show that the real Fourier series of period 2 for $g(x) = x^2$ in the range $-1 \leqslant x \leqslant 1$ is

$$g(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(\pi n x).$$

[6]

(b) By considering $\displaystyle\int_{-1}^{1} [g(x)]^2 \, dx$ calculate the sum $\displaystyle\sum_{r=1}^{\infty} r^{-4}$. [6]

(c) Find the real Fourier series of period 2 for $f(x) = \cosh x$ in the range $-1 \leqslant x \leqslant 1$ in the form

$$f(x) = \sinh(1)\left(A_0 + \sum_{n=1}^{\infty} A_n(x) \right),$$

by determining the constant A_0 and the functions $A_n(x)$. [8]

16S

(a) If $\mathbf{m} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$ is a constant vector and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector, show that

(i)

$$\boldsymbol{\nabla}\cdot\mathbf{r} = 3;$$

(ii)

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3};$$

(iii)

$$\boldsymbol{\nabla}\left(\frac{\mathbf{m}\cdot\mathbf{r}}{r^3}\right) = \frac{\mathbf{m}}{r^3} - \frac{3(\mathbf{m}\cdot\mathbf{r})}{r^5}\,\mathbf{r}.$$

[7]

(b) If $\nabla^2\phi = 0$ and $\mathbf{m}$ is a constant vector, show that

$$\boldsymbol{\nabla}\times(\mathbf{m}\times\boldsymbol{\nabla}\phi) + \boldsymbol{\nabla}(\mathbf{m}\cdot\boldsymbol{\nabla}\phi) = \mathbf{0}.$$

[6]

(c) Using Cartesian coordinates show that $\phi = 1/r$ satisfies $\nabla^2\phi = 0$ for $r \neq 0$.

Calculate $\mathbf{E} = -\boldsymbol{\nabla}\phi$ where $\phi = 1/r$.

What is the value of $\boldsymbol{\nabla}\times\mathbf{E}$? [7]

[*Notation* $\boldsymbol{\nabla}\times$ *is equivalent to* $\boldsymbol{\nabla}\wedge$]

17X

(a) State the definitions of the mean and variance of a continuous random variable X with probability density function $p(x)$ for $-\infty < x < \infty$. [2]

(b) The Normal Distribution has a probability density function with two parameters μ and σ:

$$p(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].$$

Use your definitions of the mean and variance to prove that the mean of the Normal Distribution is μ and the variance is σ^2. [7]

[*Hint: You may use* $\int_{-\infty}^{\infty} \exp[-u^2]\, du = \sqrt{\pi}$.]

(c) Find the simplest form (i.e. with fewest 'P's) for each of the following probabilities ($\bar{B}$ indicates the complement of B):

(i) $P(A|B)P(B)P(C|A \cap B)$;

(ii) $P(A|\bar{B})P(\bar{B}) + P(B)P(A|B)$;

(iii) $P(B|A)P(A)/P(A|B)$;

(iv) $P(A) + P(B) - P(A \cap B)$;

(v) $P(A \cap B)/P(A)$. [5]

(d) A standard pack of 52 cards (four suits of thirteen cards) is shuffled and five cards are drawn. Calculate the probability of drawing a "full house" (i.e. a pair and a triple, e.g. two sevens and three kings). Leave your answer in the form $\frac{N}{\binom{52}{5}}$ where N is the number to be determined. [6]

18Y

(a) Evaluate

$$\int_{y=0}^{y=1} \int_{x=0}^{x=2} x e^{xy} dx\, dy.$$

[4]

(b) By reversing the order of integration evaluate

$$\int_{y=0}^{y=\frac{1}{2}} \int_{x=-\sqrt{1-4y^2}}^{x=\sqrt{1-4y^2}} y\, dx\, dy.$$

[8]

(c) Find the volume V of the region that lies inside the quarter cylinder $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and between the planes $x + y + z = 4$ and $z = 0$, where (r, θ, z) are cylindrical polar coordinates. [8]

19R*

(a) Calculate $\boldsymbol{\nabla} \cdot \mathbf{F}$ where $\mathbf{F} = (y - x)\,\mathbf{i} + x^2 z\,\mathbf{j} + (x^2 + z)\,\mathbf{k}$. Hence use the divergence theorem to evaluate the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS},$$

where S is the open surface of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geqslant 0$. [10]

(b) Use Stokes' theorem to evaluate

$$\int_S (\,\boldsymbol{\nabla} \times \mathbf{G}) \cdot \mathbf{n} dS,$$

where

$$\mathbf{G} = z^2\,\mathbf{i} - 3xy\,\mathbf{j} + x^3 y^3\,\mathbf{k},$$

and the surface S is the part of $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$. [10]

[*Notation* $\boldsymbol{\nabla}\times$ *is equivalent to* $\boldsymbol{\nabla}\wedge$.]

20T*

(a) Given $v = (x^2 - 1)^l$, where l is a non-negative integer, show that

$$(x^2 - 1)\frac{dv}{dx} = 2lxv.$$

By differentiating this result $l + 1$ times, verify that $y = \dfrac{d^l v}{dx^l}$ satisfies the Legendre equation

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0.$$

[10]

(b) Show that

$$P_l(x) \equiv \frac{1}{2^l l!}\frac{d^l v}{dx^l}$$

is a polynomial of degree l (i.e. the highest power of x appearing in the polynomial is l). [3]

(c) Show that $P_l(1) = 1$. [7]
[*Hint: write* $v = (x + 1)^l (x - 1)^l$.]

END OF PAPER