UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday 27 May 2008 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover shett
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

Let (r, θ, ϕ) be standard spherical polar coordinates in three dimensions, satisfying the differential relation

$$d\mathbf{r} = \mathbf{e}_r \, dr + \mathbf{e}_\theta r \, d\theta + \mathbf{e}_\phi r \sin\theta \, d\phi \, .$$

Consider the vector fields defined as follows:

$$\begin{aligned}
\mathbf{A}^{(+)} &= \frac{1}{r} \tan\frac{\theta}{2} \, \mathbf{e}_\phi \qquad && (r \neq 0, \, \theta \neq \pi), \\
\mathbf{A}^{(-)} &= -\frac{1}{r} \cot\frac{\theta}{2} \, \mathbf{e}_\phi \qquad && (r \neq 0, \, \theta \neq 0), \\
\mathbf{B} &= \frac{1}{r^2} \, \mathbf{e}_r && (r \neq 0) \, .
\end{aligned}$$

(a) Give a clearly labeled sketch of the curves of constant θ and constant ϕ on the sphere $r = a$. Draw in the corresponding unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ at a point on the surface with $\theta \neq 0, \pi$; are the unit vectors well-defined at $\theta = 0$ or π? Comment briefly on the fact that $\mathbf{A}^{(+)}$ and $\mathbf{A}^{(-)}$ are well-defined at $\theta = 0$ and π, respectively.

[6]

(b) Calculate

$$\int_C \mathbf{A}^{(\pm)} \cdot d\mathbf{r}$$

where C is a circle with $r = a$, $\theta = \alpha$ and $0 \leqslant \phi \leqslant 2\pi$.

[4]

(c) Calculate $\nabla \times \mathbf{A}^{(+)}$ and $\nabla \times \mathbf{A}^{(-)}$.

[5]

(d) Evaluate

$$\int_S \mathbf{B} \cdot d\mathbf{S}$$

where S is the sphere of radius a, centre the origin. By dividing S into two parts (each with boundary C) explain how this result is related to your calculations in parts (b) and (c).

[5]

[Recall that

$$\nabla \times \mathbf{A} = \frac{1}{h_r h_\theta h_\phi} \begin{vmatrix} h_r \mathbf{e}_r & h_\theta \mathbf{e}_\theta & h_\phi \mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ h_r A_r & h_\theta A_\theta & h_\phi A_\phi \end{vmatrix}$$

where $h_u = |\partial\mathbf{r}/\partial u|$ for $u = r, \theta, \phi$.]

2A

(a) $\phi(x, y)$ is defined on a square $0 \leqslant x, y \leqslant \ell$ and obeys

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi$$

with λ constant. Find all separable solutions with $\phi = 0$ on the boundary of the square, determining the resulting values of λ in the process.

[6]

(b) Calculate constants c_{mn} such that

$$\sum_{m,n \geqslant 1} c_{mn} \sin \frac{m\pi x}{\ell} \sin \frac{n\pi y}{\ell} = 1$$

for $0 < x, y < \ell$.

[4]

(c) A two-dimensional square slab with sides of length ℓ has temperature $T(x, y; t)$ obeying

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\kappa} \frac{\partial T}{\partial t}$$

with κ a positive constant. The temperature is initially equal to T_0 throughout the slab, but at $t = 0$ the material is immersed in a heat bath so that the temperature on the boundary is zero for $t > 0$. Find $T(x, y; t)$ for $t > 0$ and show that for large t

$$T(x, y; t) \approx \frac{16 T_0}{\pi^2} \sin \frac{\pi x}{\ell} \sin \frac{\pi y}{\ell} \, e^{-2\kappa \pi^2 t / \ell^2} .$$

[10]

3A

Explain in outline the Green's function approach to solving an equation of the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = f(x)$$

where $y(x)$ with $0 < x < \infty$ is subject to certain boundary conditions.

[4]

Given that the general solution to the homogeneous problem with $f = 0$ is $y(x) = ax + b/x$, determine the Green's functions for the boundary conditions:

(i) $y(x) \to 0$ as $x \to 0$ and $x \to \infty$; or **(ii)** $y(x) \to 0$, $y'(x) \to 0$ as $x \to 0$.

[8]

Use your answers to calculate $y(x)$ explicitly with each of the boundary conditions (i) and (ii) when $f(x) = 1$ for $0 < x < 1$ and $f = 0$ otherwise.

[8]

4A

(a) Given a function $f(x)$, define its Fourier transform $\tilde{f}(k)$ and write down the inverse transform.

Let $f(x) = 1 - x$ for $0 < x < 1$ and $f(x) = 0$ for $x \geqslant 1$. Find $\tilde{f}(k)$ in the cases where

(i) $f(x)$ is an even function; **(ii)** $f(x)$ is an odd function.

[10]

(b) State and prove Parseval's Theorem. Use this to deduce that

$$\int_0^\infty \frac{\sin^4 u}{u^4}\, du = \frac{\pi}{3}.$$

[10]

[You need not discuss conditions for convergence of any integrals.]

5B

When is a 3 by 3 complex matrix diagonalisable? Let A be a 3 by 3 matrix which is not a multiple of the identity matrix such that characteristic equation of A is $(\lambda - 1)^3 = 0$. Show that A is not diagonalisable.

[5]

Diagonalise the matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and sketch the quadric surface

$$X^T A X = 1$$

where $X^T = (x, y, z)$.

[12]

Does there exist a diagonalisable 3 by 3 complex matrix with exactly two distinct eigenvalues? Give an example or show that one does not exist.

[3]

6B

What is a Hermitian matrix? Show that eigenvalues of a Hermitian matrix are real and that eigenvectors corresponding to different eigenvalues are orthogonal with respect to a standard inner product on $\mathbb{C}^n$.

[8]

Is it true that if all eigenvalues of a matrix are real then this matrix is Hermitian? Give a proof or a counter-example.

[4]

Let A, B be Hermitian matrices. Show that AB is Hermitian if and only if $AB - BA = 0$. Find a number c (real or complex) such that $AB + cBA$ is Hermitian.

[8]

7C

(a) State the *Cauchy-Riemann equations* obeyed by the real and imaginary parts, $u(x, y)$ and $v(x, y)$ of an analytic function of a complex variable $z = x + iy$.

Show that the curves $u =$ const and $v =$ const in the x, y plane intersect at right angles.

Find a complex analytic function for which these curves are, respectively, $y^2 = x^2 + \alpha$ and $xy = \beta$ for real constants α and $\beta \neq 0$.

[7]

(b) A complex function $f(z)$ which is analytic and single-valued in an annulus $a < |z - z_0| < b$ for some a and b, has a *Laurent expansion* of the form,

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n .$$

State the condition on the coefficients a_n such that,

(1) $f(z)$ has a pole of order N at $z = z_0$, or

(2) $f(z)$ has an essential singularity at $z = z_0$.

[4]

Give all singular terms (ie terms with $n < 0$) in the Laurent expansions of the following functions around the points specified,

(i) $f(z) = 1/\sinh^3(z)$ at $z = i\pi$.

(ii) $f(z) = z^N \exp(-1/z)$ at $z = \infty$.

(iii) $f(z) = \exp(-1/z)$ at $z = 0$.

[9]

8A

Consider the series

$$y(x) \;=\; x^{\sigma} \sum_{n=0}^{\infty} a_n \, x^n \;, \qquad a_0 \neq 0 \;. \tag{$*$}$$

If the series converges, show that Bessel's equation holds:

$$x^2 y'' + x\, y' + (x^2 - \nu^2)\, y \;=\; 0 \;,$$

where ν is a real constant, provided that

$$\sigma \;=\; \pm\nu \;, \qquad a_1 \;=\; 0 \;, \qquad \{\, (\sigma+n)^2 - \nu^2 \,\}\, a_n + a_{n-2} \;=\; 0 \quad \text{for} \quad n \geqslant 2 \;.$$

Show that when $\nu = 1/2$ these conditions yield two solutions which can be written in terms of trigonometric functions.

[10]

Find the most general conditions on σ and a_n for $(*)$ to satisfy Legendre's equation

$$(1 - x^2)\, y'' - 2\, x\, y' + \lambda\, (\lambda + 1)\, y \;=\; 0,$$

where λ is some real constant. Referring only to the form of the differential equation, what radius of convergence would you expect for any solution of the form $(*)$? Show that for particular values of $\lambda \geqslant 0$ which you should determine there are solutions $(*)$ which exist for all x.

[10]

9B

Derive the Euler–Lagrange equation for $y(x)$ so that the integral

$$I = \int_a^b f(x, y, y')\, dx$$

is stationary. Show that if f does not depend on x explicitly then

$$f - y' \frac{\partial f}{\partial y'}$$

is a constant.

[10]

Let $F(y)$ be a differentiable function. Consider a line element in $\mathbb{R}^3$ and show that the geodesics on the surface $\{(x, y, z) \in \mathbb{R}^3, z = F(y)\}$ are given by $(x, y(x), F(y(x)))$ where

$$\int \sqrt{1 + \left(\frac{dF(y)}{dy}\right)^2}\, dy = ax + b$$

for some constants a, b.

[10]

10B

Let y, p, q and w be real valued functions defined on $[a,b] \subset \mathbb{R}$ such that p and w are everywhere positive and

$$p(b)\,y(b)\,y'(b) - p(a)\,y(a)\,y'(a) \;=\; 0\,.$$

Let

$$F[y] \;=\; \int_a^b \left(p\Big(\frac{dy}{dx}\Big)^2 \;+\; qy^2 \right) dx\,, \quad G[y] \;=\; \int_a^b wy^2\,dx\,.$$

Show that the stationary values of

$$\Lambda[y] \;=\; \frac{F\,[y]}{G\,[y]}$$

are eigenvalues of the Sturm–Liouville eigenvalue problem

$$-\,\frac{d}{dx}\left(p(x)\,\frac{dy}{dx}\right) + q(x)\,y - \lambda\,w(x)\,y \;=\; 0 \qquad (*)$$

and that the functions which make Λ stationary are the corresponding eigenfunctions of $(*)$.

[10]

Use a trigonometric trial function to estimate the lowest eigenvalue of the equation

$$\frac{d^2y}{dx^2} \;+\; \lambda\,xy \;=0\,, \qquad y(0) \;=\; y(\pi) \;=\; 0\,.$$

What is the sign of $\lambda_e - \lambda_t$ where λ_e and λ_t are the estimate and the actual lowest eigenvalues respectively.

[10]

END OF PAPER