UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday 13 June 2007 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A comprises short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

*You may submit answers to **all** of section A, and to no more than **five** questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

*Write on **one** side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up all of your section A answer in a single bundle, with a completed blue cover sheet.

*Each section B question has a number and a letter (for example, **2Y**).*

*Section B answers must be tied up in **separate** bundles, marked **R, S, T, X, Y** or **Z** according to the letter affixed to each question. **Do not join the bundles together**. For each bundle, a blue cover sheet **must** be completed and attached to each bundle, with the appropriate letter written in the section box.*

*A **separate** yellow master cover sheet listing all the questions attempted **must** also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1. Let $\mathbf{a} = (1, -2, 1)$ and $\mathbf{b} = (1, 0, 1)$. Find a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. [1]

2. Evaluate the integrals

   (a)
$$I = \int_0^{\pi} \sin 2x \sin 4x \, \mathrm{d}x$$
[1]

   (b)
$$J = \int_0^{\pi/2} \sin x \cos x \, \mathrm{d}x.$$
[1]

3. The function
$$f(x, y) = 2x - x^2 y + y$$
is given.

   (a) Verify that the point $(x, y) = (1, 1)$ is a stationary point of $f(x, y)$. [2]

   (b) Find the other stationary point. [1]

4. A particle is confined by a potential well given by the function
$$v(x, y, z) = 3x^2 + 3y^2 + z^2\ .$$

   (a) Find the components of the force vector $\mathbf{f} = -\nabla v$ on the particle. [1]

   (b) Find $\nabla \cdot \mathbf{f}$. [1]

   (c) Find $\nabla \wedge \mathbf{f}$. [1]

5. The column vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are given. Evaluate

   (a) $\mathbf{a}^{\mathrm{T}}\mathbf{b}$ [1]

   (b) $\mathbf{a}\mathbf{b}^{\mathrm{T}}$. [1]

6. Give the modulus and the argument (either in degrees or radians) of $(1+i)(\sqrt{3}+i)$. [2]

7. Give the first two non-zero terms of the Taylor expansion at $x = 0$ of
$$\ln(1 + x)\,.$$
[2]

8. A rate process is described by the first order differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -kx^2,$$

where $k > 0$ is a rate constant. If $x = x_0$ at time $t = 0$, determine the time $t_{1/2}$ at which $x = x_0/2$. [2]

9. Two balls are picked at random without replacement from a bag containing 3 red balls and 4 green balls.

   (a) What is the probability that the balls have different colours? [2]

   (b) What is the expectation value of the number of red balls picked? [1]

SECTION B

1Y

Find in polar coordinates (r, θ) the equation for the circle S which has radius 1 and is centred at $x = 1,\ y = 0$. [3]

Find in polar coordinates the equation for the tangent T to the circle S at the point $(2, 0)$. [3]

A curve C (known as the Cissoid of Diocles) is defined as follows. Draw a straight line from the origin O which intersects the circle S (again) at point Q, and intersects the tangent T at point R. The point P on the line is defined so that $OP = QR$. As the point Q moves around the circle, the point P traces out a curve C. Find the polar equation for the curve C. [6]

Hence or otherwise show that the Cartesian equation for C is

$$y^2(2 - x) = x^3.$$

[4]

Sketch the circle S, the tangent T and the curve C. [4]

2Y

Use the substitution $t = \tan(x/2)$ to show that

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{2 + \sin x} = \int_0^1 \frac{\mathrm{d}t}{t^2 + t + 1}.$$

[10]

Hence, or otherwise, show that

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{2 + \sin x} = \frac{\pi}{3\sqrt{3}}\,.$$

[10]

3X

(a) The probability of an experiment that involves counting events having the result $N = n$ (where n is a non-negative integer) is

$$P_n = A\rho^n,$$

where ρ $(0 < \rho < 1)$ is given. Find the normalising constant A. [3]

Calculate the probability that $N > n$. [2]

Calculate the probability that $N > n$, *conditional* on $N > m$ $(n > m)$. [2]

(b) The probability density function for a continuous random variable X is

$$f(x) = B\rho^x \equiv Be^{-\lambda x}, \quad (\lambda = \ln(\rho^{-1}))$$

where x takes values between 0 and ∞. Find the normalising constant B. [3]

Calculate the probability that $X > x$, *conditional* on $X > y$ $(x > y)$. [4]

Deduce the probability density function for X, *conditional* on $X > y$. [2]

Calculate the variance of X, *conditional* on $X > y$. [4]

4Z

(a) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = e^{2x}\,.$$

[8]

(b) Find the solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 30\cos 5x\,,$$

given that $y = \mathrm{d}y/\mathrm{d}x = 0$ at $x = 0$. [8]

Sketch the solution for $x > 0$. [4]

5T

(a) Give a necessary condition for the expression

$$u(x,y)\mathrm{d}x + v(x,y)\mathrm{d}y$$

to be an exact differential. [2]

(b) Reduce the following expression to a single partial derivative:

$$\left(\frac{\partial v}{\partial u}\right)_y \left(\frac{\partial u}{\partial x}\right)_y .$$

[2]

The internal energy U of a gas can be regarded as a function of the entropy S and the volume V. It is given that

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V,$$

where T is the temperature and p is the pressure.

Show that

$$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{T}{C_V}\left(\frac{\partial p}{\partial T}\right)_V ,$$

where $C_V = \left(\frac{\partial U}{\partial T}\right)_V$. [9]

For one mole of an ideal gas, $pV = RT$, $C_V = (3/2)R$ and, when S is held constant, $p \propto V^{-5/3}$. Evaluate both sides of the above expression and confirm that they are equal. [7]

6Z

(a) A force field $\mathbf{F}$ is given in Cartesian coordinates by

$$\mathbf{F} = (2xy + z,\, x^2 + 2y,\, x)\,.$$

Find $\nabla \wedge \mathbf{F}$. [3]

(b) Find a suitable potential ψ such that $\mathbf{F} = -\nabla\psi$. [3]

(c) Evaluate $\int \mathbf{F} \cdot \mathrm{d}\mathbf{x}$ along the straight line connecting the origin to the point $(1, 1, 1)$ and verify that your result is consistent with the change in potential ψ. [4]

(d) The surface S of an ellipsoid is defined parametrically by $\mathbf{x} = (b \sin\theta \cos\phi,\, b \sin\theta \sin\phi,\, a \cos\theta)$, where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. By using

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{x}}{\partial \theta} \wedge \frac{\partial \mathbf{x}}{\partial \phi}\, \mathrm{d}\theta \mathrm{d}\phi\,,$$

evaluate directly the integral

$$\int_S \mathbf{G} \cdot \mathrm{d}\mathbf{S}$$

over the surface of the ellipsoid, where

$$\mathbf{G} = (xz^2,\, xy^2,\, z^3).$$

[*You should not use the divergence theorem.*] [10]

7S

(a) Below are statements about square matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ all having the same dimension $N \times N$. Moreover $\mathbf{A}$ and $\mathbf{B}$ are invertible.

(i) $\det\left(\mathbf{B}^{-1}\mathbf{A}\mathbf{B}\right) = \det(\mathbf{A})$ [2]

(ii) $\mathrm{Tr}(\mathbf{ABC}) = \mathrm{Tr}(\mathbf{BAC})$ [2]

(iii) $\mathbf{A}^{-1} + \mathbf{B}^{-1} = \mathbf{A}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{B}^{-1}$. [2]

Indicate which of these statements is true and which is false. If a statement is true, give a brief proof of the relation.

(b) Given

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Show that $\mathbf{C}^2 = \mathbf{C}^{-1}$. Hence, or otherwise, compute $\mathbf{C}^{16}$. [4]

(c) The matrix $\mathbf{M}$ is defined by

$$\mathbf{M} = \begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & \mu \end{pmatrix},$$

where μ is a real parameter.

(i) What condition must μ satisfy for the inverse $\mathbf{M}^{-1}$ of $\mathbf{M}$ to exist? [2]

(ii) Express $\mathbf{M}^{-1}$ as a function of the parameter μ. [4]

(d) The variables x, y and z satisfy the following set of simultaneous linear equations

$$\begin{aligned} \mu x + y &= 1 \\ x + z &= 2\,, \\ y + \mu z &= 1 \end{aligned}$$

where μ is a real parameter.

(i) Find the values of x, y and z for all nonzero values of μ. [2]

(ii) Determine the solutions of these equations when $\mu = 0$. What is their locus in Cartesian space (x, y, z)? [2]

8X

The function $f(x)$ is periodic with period 2, and

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 2x, & 0 \leqslant x < 1. \end{cases}$$

Find its Fourier series. [10]

Deduce the Fourier series of the functions $f_e(x)$ and $f_o(x)$, both periodic with period 2, with

$$f_e(x) = |x|, \quad f_o(x) = x, \quad -1 < x < 1.$$

[4]

By considering the Fourier series of $f_e(x)$ and $f_o(x)$ at suitably-chosen x, show that

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8},$$

[3]

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)} = \frac{\pi}{4}.$$

[3]

9R*

Use the method of Lagrange multipliers to find a stationary value of the function

$$f(x,y) = 2x^2 + y^2$$

along the path

$$y = (x-2)^2.$$

[8]

Show that there is only one constrained stationary point. [2]

Draw a sketch of the contours of $f(x,y)$ and superimpose a sketch of the constraint path. Include the contour that passes through the constrained stationary point and label its intersections with the coordinate axes and the constraint path. [6]

What is the relationship between the contour and the constraint path at the constrained stationary point? [2]

Use your sketch to argue that the constrained stationary point is a constrained minimum. [2]

10T*

Fluid is flowing between two parallel plates situated at $y = 0$ and $y = 1$. The driving pressure is turned off at time $t = 0$ and the fluid velocity $u(y,t)$ subsequently satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \ ,$$

where ν is a positive constant.

Suppose that u can be expressed as the product of two functions,

$$u(y,t) = Y(y)T(t) \ .$$

Show that

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = aY,$$

where a is an arbitrary constant, and find a corresponding ordinary differential equation for $T(t)$. [8]

The boundary conditions are $u(0,t) = u(1,t) = 0$ for all $t > 0$. If initially $u(y,0) = \sin(\pi y)$, find the solution for $u(y,t)$ in the interval $0 \leqslant y \leqslant 1$. [8]

State the principle of superposition for linear differential equations. Bearing this in mind, write down the solution for the initial condition $u(y,0) = \sin(\pi y) + \frac{1}{10}\sin(3\pi y)$. [4]

END OF PAPER