NATURAL SCIENCES TRIPOS Part IB
NATURAL SCIENCES TRIPOS Part II (General)

Friday 28 May 2004 9.00 to 12.00

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question will be indicated in the right-hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **6B**).

Answers must be tied up in **separate** bundles, marked **A, B or C** according to the letter affixed to each question.

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A **separate** yellow master cover sheet listing all the questions attempted **must** also be completed.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C In plane polar coordinates, (r, θ), Poisson's equation for a potential $\varphi(r, \theta)$ generated by a source $\rho(r, \theta)$ is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = \rho\,.$$

(a) Using separation of variables derive the general solution of Poisson's equation that is single-valued and finite in the domain $0 \leqslant r \leqslant a$ if

$$\rho(r, \theta) = \sum_{n=0}^{\infty} \alpha_n r^n \cos\theta\,,$$

where the α_n are known constants. [11]

Find the particular solution if $\varphi = \sin 2\theta$ on $r = a$ for the special case when $\alpha_1 = 1$ and $\alpha_n = 0$ if $n \neq 1$. [4]

(b) The velocity potential $\varphi(r, \theta)$ for inviscid flow in two dimensions satisfies Laplace's equation (i.e. Poisson's equation with $\rho = 0$). Assuming that φ is single-valued, solve for $\varphi(r, \theta)$ in $r \geqslant a$ subject to the boundary conditions

$$\frac{\partial \varphi}{\partial r} = 0 \quad \text{on} \quad r = a\,,$$

and

$$\frac{\partial \varphi}{\partial r} \to U\cos\theta \quad \text{as} \quad r \to \infty\,,$$

where U is a constant. [5]

2C The fundamental solution to Poisson's equation in three-dimensions,

$$\nabla^2 G = \delta(\mathbf{x}) \,,$$

is

$$G = -\frac{1}{4\pi r} \,,$$

where $r = |\mathbf{x}|$. By using Green's identity, which should be clearly stated, deduce that the solution of

$$\nabla^2 \psi = -\frac{q(\mathbf{x})}{\epsilon_0} \,,$$

for a charge distribution $q(\mathbf{x})$ that tends to zero as $|\mathbf{x}| \to \infty$, is given by

$$\psi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{q(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, \mathrm{d}V' \,,$$

where the integral is over all space. [6]

A total charge Q is uniformly distributed over a disc that is in the plane $z = 0$ and occupies the region $\rho \leqslant a$, where (ρ, ϕ, z) are cylindrical polar coordinates. Find the potential on the symmetry axis $\rho = 0$, being careful to state clearly the solution for both $z > 0$ and $z < 0$. Expand this solution (i) for $0 < |z| \ll a$ correct to $O(|z|^2)$, and (ii) for $|z| \gg a$ correct to $O(|z|^{-3})$, where $z = r\cos\theta$. [9]

The general three-dimensional axisymmetric solution to Laplace's equation, $\nabla^2 \Psi = 0$, can be expressed as

$$\Psi = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta) \,,$$

where (r, θ, ϕ) are spherical polar coordinates, the A_n and B_n are constants, and P_n is the Legendre polynomial of degree n (recall that $P_n(1) = 1$). Find expressions for ψ off the symmetry axis (i) for $0 < |r| \ll a$ correct to $O(r^2)$, and (ii) for $|r| \gg a$ correct to $O(r^{-3})$. [5]

3B (a) By writing $z = x + iy$, derive the Cauchy-Riemann equations for the real and imaginary parts of an analytic function $f(z) = u(x,y) + iv(x,y)$ in Cartesian coordinates (x,y), where the functions u and v are real. [4]

Show that the function $g(z) = z\,Re(z)$ is differentiable only at $z = 0$, and find $g'(0)$. [4]

(b) By writing $z = re^{i\theta}$, or otherwise, derive the Cauchy-Riemann equations for the real and imaginary parts of an analytic function $f(z) = U(r,\theta) + iV(r,\theta)$ in polar coordinates (r,θ), where the functions U and V are real. [5]

(c) Calculate the value of the integral

$$I = \int_\gamma \log z \,\mathrm{d}z\,,$$

where γ is a positively oriented closed contour, in the cases when

(i) γ is a unit circle, the branch cut is taken along the positive real axis, and $\log z$ is real on the 'upper' side of the branch cut (i.e. $\log z$ is real when $\theta = 0+$);

(ii) γ is a circle of radius R, the branch cut is taken along the positive imaginary axis, and $\log z$ is real on the positive real axis. [7]

[*You may quote the result that*

$$\int \theta\, \mathrm{e}^{i\theta}\,\mathrm{d}\theta = (1 - i\theta)\,\mathrm{e}^{i\theta}\,. \quad]$$

4B State the residue theorem for the integral of a function of a complex variable around a closed contour. [2]

The function $f(z)$ has the form

$$f(z) = \frac{\phi(z)}{\psi(z)}$$

in a neighbourhood of a simple pole $z = a$, where $\phi(z)$ and $\psi(z)$ are analytic at $z = a$, $\phi(a) \neq 0$, $\psi(a) = 0$ and $\psi'(a) \neq 0$. Show that the residue of $f(z)$ at $z = a$ is given by $\phi(a)/\psi'(a)$. Use this result to evaluate the residue of $\operatorname{cotan} z$ at $z = k\pi$, where k is an integer. [6]

Evaluate, using the residue theorem, the integral

$$I = \int_0^\infty \frac{\log x}{x^2 + a^2}\,\mathrm{d}x, \qquad a > 0.$$

What is the value of this integral at $a = 1$? [12]

[*Hint: You may find it useful to consider a large semicircular contour.*]

5C Define the convolution of the functions $f(t)$ and $g(t)$ on the assumption that the functions vanish for $t < 0$. Derive an expression for the Laplace transform of the convolution in terms of the Laplace transforms of $f(t)$ and $g(t)$. [5]

For $t \geqslant 0$ the function $y(t)$ satisfies the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y = f(t)\,.$$

Using Laplace transforms and the convolution theorem show that the general solution to this equation is

$$y(t) = y(0)\cos t + \frac{\mathrm{d}y}{\mathrm{d}t}(0)\,\sin t + \int_0^t f(\tau)\sin(t-\tau)\mathrm{d}\tau\,,$$

where Jordan's lemma may be used without proof as long as you demonstrate that the conditions of the lemma hold. Confirm that this solution satisfies both the equation, and the initial conditions at $t = 0$. [13]

Evaluate the solution in the special case when $f(t) = \delta(t - T)$, where $\delta(t)$ is the delta function and $T > 0$. [2]

6B Define an isotropic tensor. Write down the general forms of all non-zero isotropic tensors of ranks 1, 2, and 3. [3]

A vector field u_i has the following components in a particular system of Cartesian coordinates x_i:

$$u_1 = x_1 x_2^2, \quad u_2 = x_2 x_3^2, \quad u_3 = x_3 x_1^2.$$

Express the tensor $\partial u_i / \partial x_k$ as a linear combination of $\epsilon_{ijk}\omega_j$ (where ω_j is a vector to be determined) and a symmetric tensor e_{ik}. [9]

Find the directions of the principal axes of e_{ik} at the point $x_1 = 2$, $x_2 = 3$ and $x_3 = 0$, and determine the corresponding principal values. [8]

[*Note: suffix notation and the summation convention are assumed in this question.*]

[TURN OVER

7C A system is subject to small oscillations. Define the terms *normal modes*, *normal frequencies* and *normal coordinates*. [2]

Consider three masses $m_1 = m$, $m_2 = m$, and $m_3 = \mu m$ constrained to lie on the circumference of a unit circle. The masses are joined by three equal springs with spring constant k, each of which is similarly constrained to lie along arcs of the circle. At equilibrium the masses are positioned equidistant around the circumference. Let the angular displacements of the masses around the circle be θ_1, $2\pi/3 + \theta_2$ and $4\pi/3 + \theta_3$. Determine the kinetic energy and potential energy of small departures from equilibrium, and derive the equations of motion for the system. [6]

Determine the normal frequencies and normal modes of the system and interpret your results. Briefly comment on the case $\mu = 1$. [12]

8A Given a finite group G and a subgroup H, define a *right coset* of H in G. [2]

Show that every coset of H contains the same number of elements. [5]

Let M be the set of matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b and c are arbitrary integers. Show that M forms a group under matrix multiplication. [6]

Show that the subset of matrices with $a = b = 0$ forms a subgroup of M, and describe the right cosets of this subgroup. [7]

9A Give a minimal generating set for the set D of symmetries of the square. List the elements of D and find subgroups of order 2 and 4. [4]

If G and F are groups, define a homomorphism and an isomorphism from G to F. [2]

The quaternion group Q has elements

$$\{\pm 1, \pm i, \pm j, \pm k\}, \quad \text{where} \quad i^2 = j^2 = k^2 = ijk = -1\,.$$

Show that there exists a homomorphism φ to Q from a subgroup of D of order 4, and find the kernel of φ. [4]

State what is meant by a *cyclic* group. Show that a finite cyclic group G has no proper subgroup if and only if the order, $|G|$, of G is prime. [10]

10A Explain what is meant by a representation $\mathbf{D}$ of a group G, and define a faithful representation. [3]

Give a faithful 2-dimensional representation of the group $C_4 = \{1, i, -1, -i\}$, and give a geometrical interpretation. [4]

If $\mathbf{D}$ is an n-dimensional representation of a group G, and S is an invertible $n \times n$ matrix, show that the map $\widetilde{\mathbf{D}}$ defined by $\widetilde{\mathbf{D}}(g) = S\mathbf{D}(g)S^{-1}$ is also a representation, where $g \in G$. [4]

Representations $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ above are *equivalent*. Define the characters of a representation, and show that

(i) characters of equivalent representations are the same, and

(ii) elements of the same conjugacy class have the same character. [9]

[END OF PAPER