UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday 9 June 2003 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than **six** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Each question has a number and a letter (for example, **3B***).*

Answers must be tied up in **separate** *bundles, marked* **A, B, C, D, E** *or* **F** *according to the letter affixed to each question.* **Do not join the bundles together**.

For each bundle, a blue cover sheet **must** *be completed and attached to each bundle, with the appropriate letter written in the section box.*

A **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

1A The vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ form reciprocal sets, defined such that

$$\mathbf{A} = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}, \quad \mathbf{B} = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}, \quad \mathbf{C} = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})},$$

and $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \neq 0$. Show that:

(a) $\mathbf{A}\cdot\mathbf{a} = \mathbf{B}\cdot\mathbf{b} = \mathbf{C}\cdot\mathbf{c} = 1$,

(b) $\mathbf{A}\cdot\mathbf{b} = \mathbf{A}\cdot\mathbf{c} = 0$,

(c) $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are non-coplanar. [8]

Show that the vectors $\mathbf{a} = (-1,\, 1,\, 0)$, $\mathbf{b} = (0,\, 2,\, 1)$ and $\mathbf{c} = (1,\, 0,\, -1)$ are non-coplanar. Find the reciprocal basis, and hence write the vector $\mathbf{d} = (2,\, 1,\, -1)$ in terms of the basis $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. [12]

2A*

(a) Evaluate the following limits:

(i) $\dfrac{(x^2+3)(2x^2-1)}{(3x^4-x^3-1)}$ as $x \to 0$, as $x \to 1$, and as $x \to \infty$; [3]

(ii) $\left(1 - \dfrac{a}{x}\right)^x$ as $x \to \infty$.

[Hint for part (ii): first take logarithms.] [5]

(b) L'Hôpital's rule for finding limits states that, if $f(a) = g(a) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided this limit exists. Using l'Hôpital's rule, or otherwise, evaluate:

(i) $\dfrac{1}{(3x-x^3)}\displaystyle\int_0^x e^{-y^2}\,dy$ as $x \to 0$; [4]

(ii) $\dfrac{1-\sin(\pi x/2)}{x^3-3x+2}$ as $x \to 1$. [8]

3B

(a) Find the position and nature of each of the stationary points of the function

$$f(x) = \frac{x^3}{3}\, e^{-\frac{3}{2}x^2}\,.$$

[9]

(b) Sketch the function $f(x)$. [3]

(c) Evaluate $\int_{-\infty}^{\infty} f(x)dx$ and $\int_0^{\infty} f(x)dx$. [8]

4B By integrating by parts, or otherwise, evaluate

$$\int (1+3x^2)\,\ln(1+x^2)dx\,.$$

[7]

By making suitable substitutions, or otherwise, evaluate

(i) $$\int \frac{dx}{x\ \ln x}\,;$$ [6]

(ii) $$\int \frac{\sin^3 x}{\cos^4 x}\, dx\,.$$ [7]

5C

(a) Let $M = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$. Show by direct calculation that:

(i) $\det(MN) = \det(M)\ \det(N)$; [4]

(ii) $\big(MN\big)^{-1} = N^{-1}M^{-1}$. [4]

(b) Show, by means of counterexamples, that the following identities do not hold for arbitrary non-singular matrices A and B:

(i) $\det(A+B) = \det(A) + \det(B)$;

(ii) $\big[A+B\big]^{-1} = A^{-1} + B^{-1}$;

(iii) $\big(AB\big)^{-1} = A^{-1}B^{-1}$. [7]

(c) Let M be a $3 \times n$ matrix and let M^{T} denote the transpose of M. For what values of n are the following consistent with the rules of matrix manipulation:

(i) M^2 ;

(ii) $(M^{\mathrm{T}})^2$;

(iii) MM^{T} ;

(iv) $M + M^{\mathrm{T}}$;

(v) M^{-1} ? [5]

6C* Use the method of separation of variables to find the solution $U(x,t)$ of the heat equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} \qquad (t \geqslant 0,\ 0 \leqslant x \leqslant \pi)$$

that satisfies

$$U(0,t) = 0\ ; \qquad U(\pi,t) = 0\ ; \qquad U(x,0) = 3\sin x - \sin 3x\ .$$ [20]

7D The force field $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{pmatrix} \mu z + y - x \\ x - \lambda z \\ z + (\lambda - 2)y - \mu x \end{pmatrix} ,$$

where λ and μ are real parameters.

(i) Compute the line integral $\int \mathbf{F} \cdot \mathrm{d}\mathbf{x}$ along the straight line from $(0,0,0)$ to $(1,1,1)$. [6]

(ii) Compute the line integral $\int \mathbf{F} \cdot \mathrm{d}\mathbf{x}$ along the path $\mathbf{x}(t) = (t, t, t^2)$ from $(0,0,0)$ to $(1,1,1)$. [7]

(iii) Find the values of λ and μ for which $\mathbf{F}$ is conservative. [7]

8D

(a) Find, by any method, the first three non-zero terms in the Taylor expansion about $x = 0$ of the following functions:

(i) $\dfrac{\cos x}{1 - x}$; [4]

(ii) $\sin^3 x$; [6]

(iii) e^{e^x}. [4]

(b) Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Given that $f(0) \neq 0$, find the first three terms in the Taylor expansion about $x = 0$ of the function $1/f(x)$. [6]

9E

(a) Show that

$$\sinh(ix) = i \sin x$$

and that

$$\cosh(ix) = \cos x .$$

Deduce that

$$\tanh(ix) = i \tan x .$$

[6]

(b) Express $\sinh x$ and $\cosh x$ as power series in x, quoting the general term. [4]

(c) Evaluate

$$\int_0^{\pi/2} \cosh x \cos x \, dx .$$

[10]

10E*

(a) (i) By means of a diagram, show that

$$\int_5^\infty \frac{1}{(x+1)^2}\,dx < \sum_{n=6}^\infty \frac{1}{n^2} < \int_5^\infty \frac{1}{x^2}\,dx\,.$$

[4]

(ii) Given that

$$\sum_{n=1}^{5} \frac{1}{n^2} = \frac{5269}{3600}\,,$$

use the inequality above to show that

$$\frac{5869}{3600} < \sum_{n=1}^\infty \frac{1}{n^2} < \frac{5989}{3600}\,.$$

[8]

(b) Given that

$$\sum_{n=1}^{12} \frac{1}{n} = \frac{86021}{27720}\,,$$

use a similar method to evaluate

$$\sum_{n=1}^{100} \frac{1}{n}$$

to two significant figures. [8]

11F

(a) Find the general solutions of the differential equations

(i)

$$(1+x^2)\frac{dy}{dx} = ky,$$

where k is a constant, and [5]

(ii)

$$\frac{dy}{dx} - x^2 y = x^2\,.$$

[5]

(b) By writing $y(x) = x\,v(x)$, or otherwise, solve the differential equation

$$x\frac{dy}{dx} - y \;=\; \cot\left(\frac{y}{x}\right),$$

given that $y = \dfrac{\pi}{4}$ when $x = 1$. [10]

12F Sketch the region R of the xy-plane defined by

$$x \geqslant 0, \quad y \geqslant 0,$$
$$x^2 + y^2 \leqslant 1,$$
$$xy \geqslant \frac{1}{4}\,.$$

[3]

Show that, in terms of plane polar coordinates (r, θ), the two points in R at which the curves $x^2 + y^2 = 1$ and $xy = \frac{1}{4}$ intersect are located at $(r = 1,\ \theta = \pi/12)$ and $(r = 1,\ \theta = 5\pi/12)$. [4]

Show that

$$\int\int_R xy(x^2 + y^2)dxdy = \frac{\sqrt{3}}{32}\,.$$

[13]