2A Differential Equations

(a) For a differential equation of the form $\frac{dy}{dx} = f(y)$, explain how $f'(y)$ can be used to determine the stability of any equilibrium solutions and justify your answer.

---

[3] *If $f'(y_0) > 0$ then the equilibrium point y_0 is unstable, because locally $\frac{dy}{dx} \approx f'(y_0)(y - y_0)$ has solution $y \approx e^{f'(y_0)x} + y_0$, which diverges from y_0. Similarly, if $f'(y_0) < 0$ then the equilibrium point is stable.*

(b) Find the equilibrium solutions of the differential equation

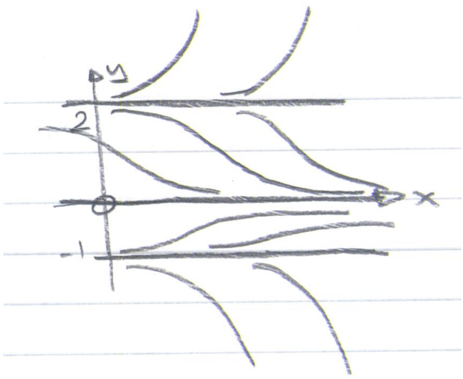
$$\frac{dy}{dx} = y^3 - y^2 - 2y$$

and determine their stability. Sketch representative solution curves in the (x, y)-plane.

---

[2] *$f(y) = y^3 - y^2 - 2y$ so $f(y) = 0$ at $y = 0$ or $y = -1$ or $y = 2$.*

[2] *$f'(y) = 3y^2 - 2y - 2$. So $f'(0) = -2$ stable, $f'(2) = 6$ unstable and $f'(-1) = 3$ unstable.*



[3]

---

[10]

7A Differential Equations

(a) Define the Wronskian W of two solutions $y_1(x)$ and $y_2(x)$ of the differential equation

$$y'' + p(x)y' + q(x)y = 0 \ , \qquad (*)$$

and state a necessary and sufficient condition for $y_1(x)$ and $y_2(x)$ to be linearly independent. Show that $W(x)$ satisfies the differential equation

$$W'(x) = -p(x)W(x) \ .$$

---

[2] $W(x) = y_1 y_2' - y_1' y_2$

[2] *Solutions are linearly independent if $W(x) \neq 0$.*

[2] $W'(x) = y_1'y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' = y_1(-p(x)y_2' - q(x)y_2) - y_2(-p(x)y_1' - q(x)y_1) = -p(x)W(x)$

(b) By evaluating the Wronskian, or otherwise, find functions $p(x)$ and $q(x)$ such that $(*)$ has solutions $y_1(x) = 1 + \cos x$ and $y_2(x) = \sin x$.

$W(x) = (1 + \cos x)\cos x - \sin x(-\sin x) = 1 + \cos x.$

[3] *So $p(x) = -W'(x)/W(x) = \sin x/(1 + \cos x)$.*

Then $-\sin x + \dfrac{\sin x}{1+\cos x}\cos x + q(x)\sin x = 0$

[3] *so* $q(x) = 1 - \cos x/(1 + \cos x) = 1/(1 + \cos x)$

What is the value of $W(\pi)$? Is there a unique solution to the differential equation for $0 \leqslant x < \infty$ with initial conditions $y(0) = 0$, $y'(0) = 1$? Why or why not?

[1] $W(\pi) = 0$

No. Using hint about $W(\pi)$, both $y = \sin x$ and

$$y = \begin{cases} \sin x & \text{for} \quad 0 \leqslant x \leqslant \pi \\ \sin + (1 + \cos x) & \text{for} \quad x > \pi \end{cases}$$

[2] *are solutions satisfying the initial conditions.*

(c) Write down a third-order differential equation with constant coefficients, such that $y_1(x) = 1+\cos x$ and $y_2(x) = \sin x$ are both solutions. Is the solution to this equation for $0 \leqslant x < \infty$ with initial conditions $y(0) = y''(0) = 0$, $y'(0) = 1$ unique? Why or why not?

[3] $y''' + y' = 0.$

Solutions are $y = 1$, $\cos x$ and $\sin x$. Since the Wronskian is now

$$W = det \begin{pmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{pmatrix} = -1 \neq 0$$

the solutions are linearly independent for all x. So the solution $y = \sin x$

[2] *obeys the initial conditions and is unique.*

[20]

8A Differential Equations

(a) The circumference y of an ellipse with semi-axes 1 and x is given by

$$y(x) = \int_0^{2\pi} \sqrt{\sin^2\theta + x^2\cos^2\theta}\,\mathrm{d}\theta\ . \qquad (*)$$

Setting $t = 1 - x^2$, find the first three terms in a series expansion of $(*)$ around $t = 0$.

$$y = \int_0^{2\pi} \sqrt{1 - t\cos^2\theta}\,\mathrm{d}\theta \approx \int_0^{2\pi} 1 - \frac{1}{2}t\cos^2\theta - \frac{1}{8}t^2\cos^4\theta\,\mathrm{d}\theta$$
$$= 2\pi - \frac{1}{2}t\pi - \frac{1}{8}t^2 \int_0^{2\pi} \cos^4\ \mathrm{d}\theta$$

and $\int_0^{2\pi} \cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta \,\mathrm{d}\theta = 2\pi$ *with* $\int_0^{2\pi}\sin^2\theta\cos^2\theta\,\mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi}\sin^2 2\theta\,\mathrm{d}\theta = \pi/4$ *so* $\int_0^{2\pi}\cos^4\theta\,\mathrm{d}\theta = 3\pi/4$.

[4] *So* $y \approx 2\pi - t\pi/2 - 3\pi t^2/32$.

(b) Euler proved that y also satisfies the differential equation

$$x(1-x^2)y'' - (1+x^2)y' + xy = 0\ .$$

Use the substitution $t = 1 - x^2$ for $x \geqslant 0$ to find a differential equation for $u(t)$, where $u(t) = y(x)$. Show that this differential equation has regular singular points at $t = 0$ and $t = 1$.

$t = 1 - x^2$ *so* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2x\dfrac{\mathrm{d}u}{\mathrm{d}t}$ *and* $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\dfrac{\mathrm{d}u}{\mathrm{d}t} + 4x^2\dfrac{\mathrm{d}^2u}{\mathrm{d}t^2}$. *So*

$$\begin{aligned} xt(-2u' + 4x^2u'') - (2-t)(-2xu') + xu &= 0 \\ 4t(1-t)u'' + 4(1-t)u' + u &= 0 \\ u'' + \frac{1}{t}u' + \frac{1}{4t(1-t)}u &= 0 \end{aligned}$$

[4]

and since $1/t$ *diverges at* $t = 0$ *and* $1/4t(1-t)$ *diverges at* $t = 1$, *we have singular points at* $t = 0$ *and* $t = 1$. *But* $tp(t)$ *and* $t^2q(t)$ *are finite at* $t = 0$, *and* $(t-1)p(t)$ *and* $(t-1)^2q(t)$ *are finite at* $t = 1$, *so these*

[2] *are regular singular points.*

Show that the indicial equation at $t = 0$ has a repeated root, and find the recurrence relation for the coefficients of the corresponding power-series solution. State the form of a second, independent solution.

Try $u = \sum_{n=0}^{\infty} a_n t^{n+\sigma}$. *Then*

$$\begin{aligned} \sum a_n\left[4(1-t)t(n+\sigma)(n+\sigma-1)t^{n+\sigma-2} + 4(1-t)(n+\sigma)t^{n+\sigma-1} + t^{n+\sigma}\right] &= 0 \\ 4a_{n+1}(n+1+\sigma)(n+\sigma) - 4a_n(n+\sigma)(n+\sigma-1) + 4a_{n+1}(n+1+\sigma) - 4a_n(n+\sigma) + a_n &= 0 \\ 4a_{n+1}(n+\sigma+1)^2 - a_n\left(4(n+\sigma)^2 - 1\right) &= 0 \end{aligned}$$

Now $a_0 \neq 0$ *and* $a_{-1} = 0$ *so take* $n = -1$ *to get* $\sigma^2 = 0$. *Repeated root*

[3] $\sigma = 0$.

[4] *The recurrence relation is then* $a_{n+1} = a_n\dfrac{4(n+\sigma)^2 - 1}{4(n+\sigma+1)^2}$ *for* $n \geqslant 0$.

[1] *Call this* u_1. *The second independent solution is* $u = u_1(t)\log t + \sum b_n t^n$

Verify that the power-series solution is consistent with your answer in (a).

Take $a_0 = 2\pi$, *then* $a_1 = a_0(-1/4)$ *and* $a_2 = a_1 3/16$. *This agrees with*

[2] *the solution above.*

---

[20]