4D Analysis I

Find the radius of convergence of each of the following power series.

(i) $\sum_{n \geqslant 1} n^2 z^n$

Known (Ratio Test): If $\sum_n a_n z^n$ is a power series with $a_n \neq 0$ for all n, and there exists $l \in [0, \infty]$ such that $|a_{n+1}/a_n| \to l$ as $n \to \infty$, then the radius of convergence is $1/l \in [0, \infty]$.

[3] *(i) Use the ratio test with $a_n = n^2$. We have for all n that $|a_{n+1}/a_n| = (1 + 1/n)^2 \to 1$ as $n \to \infty$. So radius of convergence is 1.*

(ii) $\sum_{n \geqslant 1} n^{n^{1/3}} z^n$

Known (Root Test): If $\sum_n a_n z^n$ is a power series with $\sqrt[n]{|a_n|} \to l$ as $n \to \infty$ for some $l \in [0, \infty]$, then the radius of convergence is $1/l \in [0, \infty]$.

(ii) Take $a_n = n^{n^{1/3}}$. We have for all n that

$$\sqrt[n]{|a_n|} = n^{n^{-2/3}} = e^{(\log n)/n^{2/3}}.$$

And for all $\alpha > 0$ we have

$$\frac{\log n}{n^\alpha} \to 0$$

as $n \to \infty$. (To see this, note that for all $\alpha, n > 0$ we have

$$n^\alpha = e^{\alpha \log n} \geqslant \frac{1}{2}\alpha^2 (\log n)^2.)$$

Since exp *is continuous, we get*

$$e^{(\log n)/n^{2/3}} \to 1$$

[7] *as $n \to \infty$, so by root test radius of convergence is 1.*

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[10]

10E Analysis I

State and prove the Intermediate Value Theorem.

<u>IVT:</u> If $f : [a, b] \to \mathbb{R}$ is continuous and $c \in \mathbb{R}$ is such that $f(a) \leqslant c \leqslant f(b)$ or $f(b) \leqslant c \leqslant f(a)$ then there exists $x \in [a, b]$ with $f(x) = c$.

<u>Proof:</u> WLOG $f(a) \leqslant c \leqslant f(b)$. Let $S = \{t \in [a, b] : f(t) \leqslant c\}$. This is non-empty (contains a) and bounded above (by b) so has a supremum s. We claim $f(s) = c$.

Well, if $f(s) > c$ then by continuity of f there exists $\delta > 0$ such that $[s-\delta, s] \subset [a,b]$ and $f|_{[s-\delta,s]} > c$, so $s-\delta$ is an upper bound for S which is strictly less than s—contradiction.

Similarly if $f(s) < c$ then there exists $\delta > 0$ sch that $[s, s+\delta] \subset [a,b]$ and $f|_{[s,s+\delta]} < c$ so $s+\delta \in S$, contradicting the fact that s is an upper bound for S.

[5] *Thus $f(s) = c$, so we can take $x = s$.* □

A *fixed point* of a function $f : X \to X$ is an $x \in X$ with $f(x) = x$. Prove that every continuous function $f : [0,1] \to [0,1]$ has a fixed point.

Consider the function $g : [0,1] \to \mathbb{R}$ given by $g(t) = f(t) - t$. This is continuous and satisfies $g(0) = f(0) \geqslant 0$ and $g(1) = f(1) - 1 \leqslant 0$, so
[4] *by IVT there exists $x \in [0,1]$ with $g(x) = 0$, i.e. $f(x) = x$.*

Answer the following questions with justification.

(i) Does every continuous function $f : (0,1) \to (0,1)$ have a fixed point?

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[2] (i) *No.* $f(x) = \frac{x}{2}$.

(ii) Does every continuous function $f : \mathbb{R} \to \mathbb{R}$ have a fixed point?

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[2] (ii) *No.* $f(x) = x + 1$.

(iii) Does every function $f : [0,1] \to [0,1]$ (not necessarily continuous) have a fixed point?

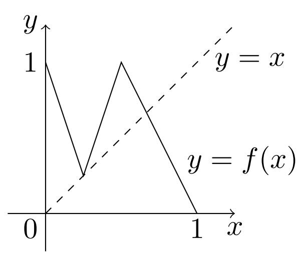
(iii) *No.*

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0. \end{cases}$$

[2]

(iv) Let $f : [0,1] \to [0,1]$ be a continuous function with $f(0) = 1$ and $f(1) = 0$. Can f have exactly two fixed points?

(iv) *Yes. Take f to be piecewise linear with the following graph*



[5]

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[20]

12D Analysis I

State and prove the Fundamental Theorem of Calculus.

FTC: If $f : [a,b] \to \mathbb{R}$ is continuous then the function $F : [a,b] \to \mathbb{R}$ defined by

$$F(t) = \int_a^t f(x)\ \mathrm{d}x$$

is differentiable on (a,b), and one-sided differentiable at a and b, with derivative f.

Proof: Fix $t \in [a,b]$. We need to show that

$$\frac{F(x) - F(t)}{x - t} \to f(t)$$

as $x \to t$ in $[a,b]$. Well, take $x \in [a,b]$ not equal to t; WLOG $t < x$. Then

$$\begin{aligned}\left|\frac{F(x) - F(t)}{x - t} - f(t)\right| &= \frac{1}{|x-t|}\left|\int_a^x f(y)\ \mathrm{d}y - \int_a^t f(y)\ \mathrm{d}y - \int_t^x f(t)\ \mathrm{d}y\right| \\ &= \frac{1}{|x-t|}\left|\int_t^x f(y) - f(t)\ \mathrm{d}y\right| \\ &\leqslant \frac{1}{|x-t|}\int_t^x \sup_{[t,x]} |f - f(t)|\ \mathrm{d}y \\ &= \sup_{[t,x]} |f - f(t)| \\ &\to 0\end{aligned}$$

as $x \to t$, since f is continuous. So F is right-differentiable on $[a,b)$
[9] *(and similarly left-differentiable on $(a,b]$) with derivative f.* $\square$

Let $f : [0,1] \to \mathbb{R}$ be integrable, and set $F(x) = \int_0^x f(t)\,\mathrm{d}t$ for $0 < x < 1$. Must F be differentiable?

No. Take

$$f(x) = \begin{cases} -1 & x \leqslant 1/2 \\ 1 & x > 1/2. \end{cases}$$

This is clearly integrable, with $F(x) = |x - 1/2| - 1/2$, which is not
[4] *differentiable at $1/2$.*

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and set $g(x) = f'(x)$ for $0 \leqslant x \leqslant 1$. Must the Riemann integral of g from 0 to 1 exist?

No. Take

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

This is differentiable at 0*, with derivative* 0*, since* $|f(x)| \leqslant |x|^2$ *for all* x *(so* $|f(x)/x| \to 0$ *as* $x \to 0$*). Clearly differentiable away from* 0*, with derivative*

$$g(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$$

by the chain rule. This is unbounded on $(0, 1]$*, so not Riemann*
[7] *integrable.*

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[20]