UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 6 June, 2019 9:00 am to 12:00 pm MAT2

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, ..., ***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet
Script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Let $f = (a, b, c)$ be a positive definite binary quadratic form with integer coefficients. What does it mean to say that f is *reduced*? Show that if f is reduced and has discriminant d, then $|b| \leqslant a \leqslant \sqrt{|d|/3}$ and $b \equiv d \pmod 2$. Deduce that for fixed $d < 0$, there are only finitely many reduced f of discriminant d.

Find all reduced positive definite binary quadratic forms of discriminant -15.

2H Topics in Analysis

State *Nash's theorem* for a non zero-sum game in the case of two players with two choices.

The role playing game Tixerb involves two players. Before the game begins, each player i chooses a p_i with $0 \leqslant p_i \leqslant 1$ which they announce. They may change their choice as many times as they wish, but, once the game begins, no further changes are allowed. When the game starts, player i becomes a Dark Lord with probability p_i and a harmless peasant with probability $1 - p_i$. If one player is a Dark Lord and the other a peasant the Lord gets 2 points and the peasant -2. If both are peasants they get 1 point each, if both Lords they get $-U$ each. Show that there exists a U_0, to be found, such that, if $U > U_0$ there will be three choices of (p_1, p_2) for which neither player can increase the expected value of their outcome by changing their choice unilaterally, but, if $U_0 > U$, there will only be one. Find the appropriate (p_1, p_2) in each case.

3G Coding and Cryptography

What does it mean to transmit reliably at rate R through a binary symmetric channel (BSC) with error probability p?

Assuming Shannon's second coding theorem (also known as Shannon's noisy coding theorem), compute the supremum of all possible reliable transmission rates of a BSC. Describe qualitatively the behaviour of the capacity as p varies. Your answer should address the following cases,

(i) p is small,

(ii) $p = 1/2$,

(iii) $p > 1/2$.

4H Automata and Formal Languages

(a) Define what it means for a context-free grammar (CFG) to be in *Chomsky normal form* (CNF). Can a CFG in CNF ever define a language containing ϵ? If G_{Chom} denotes the result of converting an arbitrary CFG G into one in CNF, state the relationship between $\mathcal{L}(G)$ and $\mathcal{L}(G_{\text{Chom}})$.

(b) Let G be a CFG in CNF. Give an algorithm that, on input of any word v on the terminals of G, decides if $v \in \mathcal{L}(G)$ or not. Explain why your algorithm works.

(c) Convert the following CFG G into a grammar in CNF:

$$S \to \; Sbb \mid aS \mid T$$

$$T \to cc$$

Does $\mathcal{L}(G) = \mathcal{L}(G_{\text{Chom}})$ in this case? Justify your answer.

5J Statistical Modelling

(a) For a given model with likelihood $L(\beta), \beta \in \mathbb{R}^p$, define the Fisher information matrix in terms of the Hessian of the log-likelihood.

Consider a generalised linear model with design matrix $X \in \mathbb{R}^{n \times p}$, output variables $y \in \mathbb{R}^n$, a bijective link function, mean parameters $\mu = (\mu_1, \ldots, \mu_n)$ and dispersion parameters $\sigma_1^2 = \ldots = \sigma_n^2 = \sigma^2$. Assume σ^2 is known.

(b) State the form of the log-likelihood.

(c) For the canonical link, show that when the parameter σ^2 is known, the Fisher information matrix is equal to

$$\sigma^{-2} X^T W X,$$

for a diagonal matrix W depending on the means μ. Identify W.

6C Mathematical Biology

A model of wound healing in one spatial dimension is given by

$$\frac{\partial S}{\partial t} = rS(1-S) + D\,\frac{\partial^2 S}{\partial x^2}\,,$$

where $S(x,t)$ gives the density of healthy tissue at spatial position x at time t and r and D are positive constants.

By setting $S(x,t) = f(\xi)$ where $\xi = x - ct$, seek a steady travelling wave solution where $f(\xi)$ tends to one for large negative ξ and tends to zero for large positive ξ. By linearising around the leading edge, where $f \approx 1$, find the possible wave speeds c of the system. Assuming that the full nonlinear system will settle to the slowest possible speed, express the wave speed as a function of D and r.

Consider now a situation where the tissue is destroyed in some window of length W, i.e. $S(x,0) = 0$ for $0 < x < W$ for some constant $W > 0$ and $S(x,0)$ is equal to one elsewhere. Explain what will happen for subsequent times, illustrating your answer with sketches of $S(x,t)$. Determine approximately how long it will take for this wound to heal (in the sense that S is close to one everywhere).

7A Further Complex Methods

The equation

$$zw'' + w = 0$$

has solutions of the form

$$w(z) = \int_\gamma e^{zt} f(t)dt,$$

for suitably chosen contours γ and some suitable function $f(t)$.

(a) Find $f(t)$ and determine the required condition on γ, which you should express in terms of z and t.

(b) Use the result of part (a) to specify a possible contour with the help of a clearly labelled diagram.

8E Classical Dynamics

A simple harmonic oscillator of mass m and spring constant k has the equation of motion

$$m\ddot{x} = -kx\,.$$

(a) Describe the orbits of the system in phase space. State how the action I of the oscillator is related to a geometrical property of the orbits in phase space. Derive the action–angle variables (θ, I) and give the form of the Hamiltonian of the oscillator in action–angle variables.

(b) Suppose now that the spring constant k varies in time. Under what conditions does the theory of adiabatic invariance apply? Assuming that these conditions hold, identify an adiabatic invariant and determine how the energy and amplitude of the oscillator vary with k in this approximation.

9B Cosmology

Consider a spherically symmetric distribution of mass with density $\rho(r)$ at distance r from the centre. Derive the pressure support equation that the pressure $P(r)$ has to satisfy for the system to be in static equilibrium.

Assume now that the mass density obeys $\rho(r) = Ar^2P(r)$, for some positive constant A. Determine whether or not the system has a stable solution corresponding to a star of finite radius.

10D Quantum Information and Computation

Let B_n denote the set of all n-bit strings and write $N = 2^n$. Let I denote the identity operator on n qubits and for $G = \{x_1, x_2, \ldots, x_k\} \subset B_n$ introduce the n-qubit operator

$$Q = -H_n I_0 H_n I_G$$

where $H_n = H \otimes \ldots \otimes H$ is the Hadamard operation on each of the n qubits, and I_0 and I_G are given by

$$I_0 = I - 2\,|00\ldots0\rangle\,\langle 00\ldots0| \qquad I_G = I - 2\sum_{x\in G} |x\rangle\,\langle x|\,.$$

Also introduce the states

$$|\psi_0\rangle = \frac{1}{\sqrt{N}}\sum_{x\in B_n}|x\rangle \qquad |\psi_G\rangle = \frac{1}{\sqrt{k}}\sum_{x\in G}|x\rangle \qquad |\psi_B\rangle = \frac{1}{\sqrt{N-k}}\sum_{x\notin G}|x\rangle\,.$$

Let $\mathcal{P}$ denote the real span of $|\psi_0\rangle$ and $|\psi_G\rangle$.

(a) Show that Q maps $\mathcal{P}$ to itself, and derive a geometrical interpretation of the action of Q on $\mathcal{P}$, stating clearly any results from Euclidean geometry that you use.

(b) Let $f : B_n \to B_1$ be the Boolean function such that $f(x) = 1$ iff $x \in G$. Suppose that $k = N/4$. Show that we can obtain an $x \in G$ with certainty by using just one application of the standard quantum oracle U_f for f (together with other operations that are independent of f).

SECTION II

11I Number Theory

Let $p > 2$ be a prime.

(a) What does it mean to say that an integer g is a primitive root mod p?

(b) Let k be an integer with $0 \leqslant k < p-1$. Let

$$S_k = \sum_{x=0}^{p-1} x^k.$$

Show that $S_k \equiv 0 \pmod{p}$. [Recall that by convention $0^0 = 1$.]

(c) Let $f(X, Y, Z) = aX^2 + bY^2 + cZ^2$ for some a, b, $c \in \mathbb{Z}$, and let $g = 1 - f^{p-1}$. Show that for any x, y, $z \in \mathbb{Z}$, $g(x, y, z) \equiv 0$ or $1 \pmod{p}$, and that

$$\sum_{x,y,z \in \{0,1,\ldots,p-1\}} g(x, y, z) \equiv 0 \pmod{p}.$$

Hence show that there exist integers x, y, z, not all divisible by p, such that $f(x, y, z) \equiv 0 \pmod{p}$.

12H Automata and Formal Languages

(a) State the *s-m-n theorem* and the *recursion theorem.*

(b) State and prove *Rice's theorem.*

(c) Show that if $g : \mathbb{N}_0^2 \to \mathbb{N}_0$ is partial recursive, then there is some $e \in \mathbb{N}_0$ such that

$$f_{e,1}(y) = g(e, y) \quad \forall y \in \mathbb{N}_0.$$

(d) Show there exists some $m \in \mathbb{N}_0$ such that W_m has exactly m^2 elements.

(e) Given $n \in \mathbb{N}_0$, is it possible to compute whether or not the number of elements of W_n is a (finite) perfect square? Justify your answer.

[*In this question* $\mathbb{N}_0$ *denotes the set of non-negative integers. Any use of Church's thesis in your answers should be explicitly stated.*]

13C Mathematical Biology

(a) A stochastic birth-death process has a master equation given by

$$\frac{dp_n}{dt} = \lambda(p_{n-1} - p_n) + \beta\left[(n+1)p_{n+1} - np_n\right] ,$$

where $p_n(t)$ is the probability that there are n individuals in the population at time t for $n = 0, 1, 2, \ldots$ and $p_n = 0$ for $n < 0$.

(i) Give a brief interpretation of λ and β.

(ii) Derive an equation for $\frac{\partial \phi}{\partial t}$, where ϕ is the generating function

$$\phi(s,t) = \sum_{n=0}^{\infty} s^n p_n(t).$$

(iii) Assuming that the generating function ϕ takes the form

$$\phi(s,t) = e^{(s-1)\, f(t)} ,$$

find $f(t)$ and hence show that, as $t \to \infty$, both the mean $\langle n \rangle$ and variance σ^2 of the population size tend to constant values, which you should determine.

(b) Now suppose an extra process is included: k individuals are added to the population at rate $\epsilon(n)$.

(i) Write down the new master equation, and explain why, for $k > 1$, the approach used in part (a) will fail.

(ii) By working with the master equation directly, find a differential equation for the rate of change of the mean population size $\langle n \rangle$.

(iii) Now take $\epsilon(n) = an + b$ for positive constants a and b. Show that for $\beta > ak$ the mean population size tends to a constant, which you should determine. Briefly describe what happens for $\beta < ak$.

14B Cosmology

[You may work in units of the speed of light, so that $c = 1$.]

Consider the process where protons and electrons combine to form neutral hydrogen atoms;

$$p^+ + e^- \leftrightarrow H^0 + \gamma.$$

Let n_p, n_e and n_H denote the number densities for protons, electrons and hydrogen atoms respectively. The ionization energy of hydrogen is denoted I. State and derive *Saha's equation* for the ratio $n_e n_p / n_H$, clearly describing the steps required.

[You may use without proof the following formula for the equilibrium number density of a non-relativistic species a with g_a degenerate states of mass m at temperature T such that $k_B T \ll m$,

$$n_a = g_a \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} \exp\left([\mu - m] / k_B T \right) ,$$

where μ is the chemical potential and k_B and h are the Boltzmann and Planck constants respectively.]

The photon number density n_γ is given as

$$n_\gamma = \frac{16\pi}{h^3} \zeta(3) \left(k_B T \right)^3 ,$$

where $\zeta(3) \simeq 1.20$. Consider now the fractional ionization $X_e = n_e/(n_e + n_H)$. In our universe $n_e + n_H = n_p + n_H \simeq \eta n_\gamma$ where η is the baryon-to-photon number ratio. Find an expression for the ratio

$$\frac{(1 - X_e)}{X_e^2}$$

in terms of $k_B T$, η, I and the particle masses. One might expect neutral hydrogen to form at a temperature given by $k_B T \sim I \sim 13$ eV, but instead in our universe it forms at the much lower temperature $k_B T \sim 0.3$ eV. Briefly explain why this happens. Estimate the temperature at which neutral hydrogen would form in a hypothetical universe with $\eta = 1$. Briefly explain your answer.

15D Quantum Information and Computation

Let $\mathcal{H}_d$ denote a d-dimensional state space with orthonormal basis $\{|y\rangle : y \in \mathbb{Z}_d\}$. For any $f : \mathbb{Z}_m \to \mathbb{Z}_n$ let U_f be the operator on $\mathcal{H}_m \otimes \mathcal{H}_n$ defined by

$$U_f |x\rangle |y\rangle = |x\rangle |y + f(x) \bmod n\rangle$$

for all $x \in \mathbb{Z}_m$ and $y \in \mathbb{Z}_n$.

(a) Define QFT, the quantum Fourier transform mod d (for any chosen d).

(b) Let S on $\mathcal{H}_d$ (for any chosen d) denote the operator defined by

$$S |y\rangle = |y + 1 \bmod d\rangle$$

for $y \in \mathbb{Z}_d$. Show that the Fourier basis states $|\xi_x\rangle = QFT |x\rangle$ for $x \in \mathbb{Z}_d$ are eigenstates of S. By expressing U_f in terms of S find a basis of eigenstates of U_f and determine the corresponding eigenvalues.

(c) Consider the following oracle promise problem:
Input: an oracle for a function $f : \mathbb{Z}_3 \to \mathbb{Z}_3$.
Promise: f has the form $f(x) = sx + t$ where s and t are unknown coefficients (and with all arithmetic being mod 3).
Problem: Determine s with certainty.

Can this problem be solved by a single query to a classical oracle for f (and possible further processing independent of f)? Give a reason for your answer.

Using the results of part (b) or otherwise, give a quantum algorithm for this problem that makes just one query to the quantum oracle U_f for f.

(d) For any $f : \mathbb{Z}_3 \to \mathbb{Z}_3$, let $f_1(x) = f(x + 1)$ and $f_2(x) = -f(x)$ (all arithmetic being mod 3). Show how U_{f_1} and U_{f_2} can each be implemented with one use of U_f together with other unitary gates that are independent of f.

(e) Consider now the oracle problem of the form in part (c) except that now f is a quadratic function $f(x) = ax^2 + bx + c$ with unknown coefficients a, b, c (and all arithmetic being mod 3), and the problem is to determine the coefficient a with certainty. Using the results of part (d) or otherwise, give a quantum algorithm for this problem that makes just two queries to the quantum oracle for f.

16I Logic and Set Theory

Define the *von Neumann hierarchy* of sets V_α. Show that each V_α is transitive, and explain why $V_\alpha \subset V_\beta$ whenever $\alpha \leqslant \beta$. Prove that every set x is a member of some V_α.

Which of the following are true and which are false? Give proofs or counterexamples as appropriate. [You may assume standard properties of rank.]

(i) If the rank of a set x is a (non-zero) limit then x is infinite.

(ii) If the rank of a set x is countable then x is countable.

(iii) If every finite subset of a set x has rank at most α then x has rank at most α.

(iv) For every ordinal α there exists a set of rank α.

17G Graph Theory

(a) What does it mean to say that a graph is *bipartite*?

(b) Show that G is bipartite if and only if it contains no cycles of odd length.

(c) Show that if G is bipartite then

$$\frac{\mathrm{ex}\,(n;G)}{\binom{n}{2}} \quad\to\quad 0$$

as $n \to \infty$.
[You may use without proof the Erdős–Stone theorem provided it is stated precisely.]

(d) Let G be a graph of order n with m edges. Let U be a random subset of $V(G)$ containing each vertex of G independently with probability $\frac{1}{2}$. Let X be the number of edges with precisely one vertex in U. Find, with justification, $\mathbb{E}(X)$, and deduce that G contains a bipartite subgraph with at least $\frac{m}{2}$ edges.

By using another method of choosing a random subset of $V(G)$, or otherwise, show that if n is even then G contains a bipartite subgraph with at least $\frac{mn}{2(n-1)}$ edges.

18F Galois Theory

Let k be a field. For m a positive integer, consider $X^m - 1 \in k[X]$, where either char $k = 0$, or char $k = p$ with p not dividing m; explain why the polynomial has distinct roots in a splitting field.

For m a positive integer, define the mth *cyclotomic polynomial* $\Phi_m \in \mathbb{C}[X]$ and show that it is a monic polynomial in $\mathbb{Z}[X]$. Prove that Φ_m is irreducible over $\mathbb{Q}$ for all m. [*Hint: If* $\Phi_m = fg$, *with* $f, g \in \mathbb{Z}[X]$ *and* f *monic irreducible with* $0 < \deg f < \deg \Phi_m$, *and* ε *is a root of* f, *show first that* ε^p *is a root of* f *for any prime* p *not dividing* m.]

Let $F = X^8 + X^7 - X^5 - X^4 - X^3 + X + 1 \in \mathbb{Z}[X]$; by considering the product $(X^2 - X + 1)F$, or otherwise, show that F is irreducible over $\mathbb{Q}$.

19I Representation Theory

In this question all representations are complex and G is a finite group.

(a) State and prove *Mackey's theorem.* State the *Frobenius reciprocity theorem.*

(b) Let X be a finite G-set and let $\mathbb{C}X$ be the corresponding permutation representation. Pick any orbit of G on X: it is isomorphic as a G-set to G/H for some subgroup H of G. Write down the character of $\mathbb{C}(G/H)$.

(i) Let $\mathbb{C}_G$ be the trivial representation of G. Show that $\mathbb{C}X$ may be written as a direct sum

$$\mathbb{C}X = \mathbb{C}_G \oplus V$$

for some representation V.

(ii) Using the results of (a) compute the character inner product $\langle 1_H \uparrow^G, 1_H \uparrow^G \rangle_G$ in terms of the number of (H, H) double cosets.

(iii) Now suppose that $|X| \geqslant 2$, so that $V \neq 0$. By writing $\mathbb{C}(G/H)$ as a direct sum of irreducible representations, deduce from (ii) that the representation V is irreducible if and only if G acts 2-transitively. In that case, show that V is not the trivial representation.

20F Algebraic Topology

Let K be a simplicial complex, and L a subcomplex. As usual, $C_k(K)$ denotes the group of k-chains of K, and $C_k(L)$ denotes the group of k-chains of L.

(a) Let

$$C_k(K, L) = C_k(K)/C_k(L)$$

for each integer k. Prove that the boundary map of K descends to give $C_\bullet(K, L)$ the structure of a chain complex.

(b) The *homology groups of K relative to L*, denoted by $H_k(K, L)$, are defined to be the homology groups of the chain complex $C_\bullet(K, L)$. Prove that there is a long exact sequence that relates the homology groups of K relative to L to the homology groups of K and the homology groups of L.

(c) Let D_n be the closed n-dimensional disc, and S^{n-1} be the $(n-1)$-dimensional sphere. Exhibit simplicial complexes K_n and subcomplexes L_{n-1} such that $D_n \cong |K_n|$ in such a way that $|L_{n-1}|$ is identified with S^{n-1}.

(d) Compute the relative homology groups $H_k(K_n, L_{n-1})$, for all integers $k \geqslant 0$ and $n \geqslant 2$ where K_n and L_{n-1} are as in (c).

21H Linear Analysis

(a) Let X be a Banach space and consider the open unit ball $B = \{x \in X : \|x\| < 1\}$. Let $T : X \to X$ be a bounded operator. Prove that $\overline{T(B)} \supset B$ implies $T(B) \supset B$.

(b) Let P be the vector space of all polynomials in one variable with real coefficients. Let $\|\cdot\|$ be any norm on P. Show that $(P, \|\cdot\|)$ is not complete.

(c) Let $f : \mathbb{C} \to \mathbb{C}$ be entire, and assume that for every $z \in \mathbb{C}$ there is n such that $f^{(n)}(z) = 0$ where $f^{(n)}$ is the n-th derivative of f. Prove that f is a polynomial.

[You may use that an entire function vanishing on an open subset of $\mathbb{C}$ must vanish everywhere.]

(d) A Banach space X is said to be uniformly convex if for every $\varepsilon \in (0, 2]$ there is $\delta > 0$ such that for all $x, y \in X$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geqslant \varepsilon$, one has $\|(x + y)/2\| \leqslant 1 - \delta$. Prove that ℓ^2 is uniformly convex.

22H Analysis of Functions

(a) Prove that in a finite-dimensional normed vector space the weak and strong topologies coincide.

(b) Prove that in a normed vector space X, a weakly convergent sequence is bounded. [Any form of the Banach–Steinhaus theorem may be used, as long as you state it clearly.]

(c) Let ℓ^1 be the space of real-valued absolutely summable sequences. Suppose (a^k) is a weakly convergent sequence in ℓ^1 which does not converge strongly. Show there is a constant $\varepsilon > 0$ and a sequence (x^k) in ℓ^1 which satisfies $x^k \rightharpoonup 0$ and $\|x^k\|_{\ell^1} \geqslant \varepsilon$ for all $k \geqslant 1$.

With (x^k) as above, show there is some $y \in \ell^\infty$ and a subsequence (x^{k_n}) of (x^k) with $\langle x^{k_n}, y\rangle \geqslant \varepsilon/3$ for all n. Deduce that every weakly convergent sequence in ℓ^1 is strongly convergent.

[*Hint: Define* y *so that* $y_i = \text{sign } x_i^{k_n}$ *for* $b_{n-1} < i \leqslant b_n$*, where the sequence of integers* b_n *should be defined inductively along with* x^{k_n}*.*]

(d) Is the conclusion of part (c) still true if we replace ℓ^1 by $L^1([0, 2\pi])$?

23F Riemann Surfaces

Let Λ be a lattice in $\mathbb{C}$, and $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ a holomorphic map of complex tori. Show that f lifts to a linear map $F : \mathbb{C} \to \mathbb{C}$.

Give the definition of $\wp(z) := \wp_\Lambda(z)$, the *Weierstrass* $\wp$*-function* for Λ. Show that there exist constants g_2, g_3 such that

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Suppose $f \in \operatorname{Aut}(\mathbb{C}/\Lambda)$, that is, $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ is a biholomorphic group homomorphism. Prove that there exists a lift $F(z) = \zeta z$ of f, where ζ is a root of unity for which there exist $m, n \in \mathbb{Z}$ such that $\zeta^2 + m\zeta + n = 0$.

24F Algebraic Geometry

Let $W \subseteq \mathbb{A}^2$ be the curve defined by the equation $y^3 = x^4 + 1$ over the complex numbers $\mathbb{C}$, and let $X \subseteq \mathbb{P}^2$ be its closure.

(a) Show X is smooth.

(b) Determine the ramification points of the map $X \to \mathbb{P}^1$ defined by

$$(x : y : z) \mapsto (x : z).$$

Using this, determine the Euler characteristic and genus of X, stating clearly any theorems that you are using.

(c) Let $\omega = \frac{dx}{y^2} \in \mathcal{K}_X$. Compute $\nu_p(\omega)$ for all $p \in X$, and determine a basis for $\mathcal{L}(\mathcal{K}_X)$.

25H Differential Geometry

(a) Let $\alpha : (a, b) \to \mathbb{R}^2$ be a regular curve without self intersection given by $\alpha(v) = (f(v), g(v))$ with $f(v) > 0$ for $v \in (a, b)$.

Consider the local parametrisation given by

$$\phi : (0, 2\pi) \times (a, b) \to \mathbb{R}^3,$$

where $\phi(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$.

(i) Show that the image $\phi((0, 2\pi) \times (a, b))$ defines a regular surface S in $\mathbb{R}^3$.

(ii) If $\gamma(s) = \phi(u(s), v(s))$ is a geodesic in S parametrised by arc length, then show that $f(v(s))^2 u'(s)$ is constant in s. If $\theta(s)$ denotes the angle that the geodesic makes with the parallel $S \cap \{z = g(v(s))\}$, then show that $f(v(s))\cos\theta(s)$ is constant in s.

(b) Now assume that $\alpha(v) = (f(v), g(v))$ extends to a smooth curve $\alpha : [a, b] \to \mathbb{R}^2$ such that $f(a) = 0$, $f(b) = 0$, $f'(a) \neq 0$, $f'(b) \neq 0$. Let $\overline{S}$ be the closure of S in $\mathbb{R}^3$.

(i) State a necessary and sufficient condition on $\alpha(v)$ for $\overline{S}$ to be a compact regular surface. Justify your answer.

(ii) If $\overline{S}$ is a compact regular surface, and $\gamma : (-\infty, \infty) \to \overline{S}$ is a geodesic, show that there exists a non-empty open subset $U \subset \overline{S}$ such that $\gamma((-\infty, \infty)) \cap U = \emptyset$.

26K Probability and Measure

(a) Let X and Y be real random variables such that $\mathbb{E}[f(X)] = \mathbb{E}[f(Y)]$ for every compactly supported continuous function f. Show that X and Y have the same law.

(b) Given a real random variable Z, let $\varphi_Z(s) = \mathbb{E}(e^{isZ})$ be its characteristic function. Prove the identity

$$\iint g(\varepsilon s) f(x) e^{-isx} \varphi_Z(s) ds\ dx = \int \hat{g}(t)\ \mathbb{E}[f(Z - \varepsilon t)] dt$$

for real $\varepsilon > 0$, where is f is continuous and compactly supported, and where g is a Lebesgue integrable function such that $\hat{g}$ is also Lebesgue integrable, where

$$\hat{g}(t) = \int g(x) e^{itx} dx$$

is its Fourier transform. Use the above identity to derive a formula for $\mathbb{E}[f(Z)]$ in terms of φ_Z, and recover the fact that φ_Z determines the law of Z uniquely.

(c) Let X and Y be bounded random variables such that $\mathbb{E}(X^n) = \mathbb{E}(Y^n)$ for every positive integer n. Show that X and Y have the same law.

(d) The Laplace transform $\psi_Z(s)$ of a non-negative random variable Z is defined by the formula

$$\psi_Z(s) = \mathbb{E}(e^{-sZ})$$

for $s \geqslant 0$. Let X and Y be (possibly unbounded) non-negative random variables such that $\psi_X(s) = \psi_Y(s)$ for all $s \geqslant 0$. Show that X and Y have the same law.

(e) Let

$$f(x;k) = 1_{\{x>0\}} \frac{1}{k!} x^k e^{-x}$$

where k is a non-negative integer and $1_{\{x>0\}}$ is the indicator function of the interval $(0, +\infty)$.

Given non-negative integers $k_1, \ldots, k_n$, suppose that the random variables $X_1, \ldots, X_n$ are independent with X_i having density function $f(\cdot; k_i)$. Find the density of the random variable $X_1 + \cdots + X_n$.

27K Applied Probability

(a) What does it mean to say that a continuous-time Markov chain $X = (X_t : 0 \leqslant t \leqslant T)$ with state space S is *reversible in equilibrium*? State the *detailed balance equations*, and show that any probability distribution on S satisfying them is invariant for the chain.

(b) Customers arrive in a shop in the manner of a Poisson process with rate $\lambda > 0$. There are s servers, and capacity for up to N people waiting for service. Any customer arriving when the shop is full (in that the total number of customers present is $N+s$) is not admitted and never returns. Service times are exponentially distributed with parameter $\mu > 0$, and they are independent of one another and of the arrivals process. Describe the number X_t of customers in the shop at time t as a Markov chain.

Calculate the invariant distribution π of $X = (X_t : t \geqslant 0)$, and explain why π is the unique invariant distribution. Show that X is reversible in equilibrium.

[*Any general result from the course may be used without proof, but must be stated clearly.*]

28J Principles of Statistics

We consider the exponential model $\{f(\cdot, \theta) : \theta \in (0, \infty)\}$, where

$$f(x, \theta) = \theta e^{-\theta x} \quad \text{for } x \geqslant 0\,.$$

We observe an i.i.d. sample $X_1, \ldots, X_n$ from the model.

(a) Compute the maximum likelihood estimator $\hat{\theta}_{MLE}$ for θ. What is the limit in distribution of $\sqrt{n}(\hat{\theta}_{MLE} - \theta)$?

(b) Consider the Bayesian setting and place a Gamma$(\alpha, \beta), \alpha, \beta > 0$, prior for θ with density

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp(-\beta\theta) \quad \text{for } \theta > 0\,,$$

where Γ is the Gamma function satisfying $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for all $\alpha > 0$. What is the posterior distribution for θ? What is the Bayes estimator $\hat{\theta}_\pi$ for the squared loss?

(c) Show that the Bayes estimator is consistent. What is the limiting distribution of $\sqrt{n}(\hat{\theta}_\pi - \theta)$?

[*You may use results from the course, provided you state them clearly.*]

29K Stochastic Financial Models

In the Black–Scholes model the price $\pi(C)$ at time 0 for a European option of the form $C = f(S_T)$ with maturity $T > 0$ is given by

$$\pi(C) = e^{-rT} \int_{-\infty}^{\infty} f\Big(S_0 \exp\big(\sigma\sqrt{T}y + (r - \tfrac{1}{2}\sigma^2)T\big)\Big) \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy.$$

(a) Find the price at time 0 of a European call option with maturity $T > 0$ and strike price $K > 0$ in terms of the standard normal distribution function. Derive the put-call parity to find the price of the corresponding European put option.

(b) The digital call option with maturity $T > 0$ and strike price $K > 0$ has payoff given by

$$C_{\text{digCall}} = \begin{cases} 1 & \text{if } S_T \geqslant K, \\ 0 & \text{otherwise.} \end{cases}$$

What is the value of the option at any time $t \in [0, T]$? Determine the number of units of the risky asset that are held in the hedging strategy at time t.

(c) The digital put option with maturity $T > 0$ and strike price $K > 0$ has payoff

$$C_{\text{digPut}} = \begin{cases} 1 & \text{if } S_T < K, \\ 0 & \text{otherwise.} \end{cases}$$

Find the put-call parity for digital options and deduce the Black–Scholes price at time 0 for a digital put.

30A Asymptotic Methods

(a) State *Watson's lemma* for the case when all the functions and variables involved are real, and use it to calculate the asymptotic approximation as $x \to \infty$ for the integral I, where

$$I = \int_0^\infty e^{-xt} \sin(t^2)\, dt.$$

(b) The Bessel function $J_\nu(z)$ of the first kind of order ν has integral representation

$$J_\nu(z) = \frac{1}{\Gamma(\nu + \frac{1}{2})\sqrt{\pi}} \left(\frac{z}{2}\right)^\nu \int_{-1}^{1} e^{izt}(1-t^2)^{\nu - 1/2}\, dt\ ,$$

where Γ is the Gamma function, $\mathrm{Re}(\nu) > 1/2$ and z is in general a complex variable. The complex version of Watson's lemma is obtained by replacing x with the complex variable z, and is valid for $|z| \to \infty$ and $|\arg(z)| \leqslant \pi/2 - \delta < \pi/2$, for some δ such that $0 < \delta < \pi/2$. Use this version to derive an asymptotic expansion for $J_\nu(z)$ as $|z| \to \infty$. For what values of $\arg(z)$ is this approximation valid?

[*Hint: You may find the substitution* $t = 2\tau - 1$ *useful.*]

31E Dynamical Systems

Consider a dynamical system of the form

$$\begin{aligned} \dot{x} &= x(1 - y + ax)\,, \\ \dot{y} &= ry(-1 + x - by)\,, \end{aligned}$$

on $\Lambda = \{(x, y) : x > 0 \text{ and } y > 0\}$, where a, b and r are real constants and $r > 0$.

(a) For $a = b = 0$, by considering a function of the form $V(x, y) = f(x) + g(y)$, show that all trajectories in Λ are either periodic orbits or a fixed point.

(b) Using the same V, show that no periodic orbits in Λ persist for small a and b if $ab < 0$.

[*Hint: for* $a = b = 0$ *on the periodic orbits with period* T*, show that* $\int_0^T (1 - x)dt = 0$ *and hence that* $\int_0^T x(1 - x)dt = \int_0^T \left[-(1 - x)^2 + (1 - x)\right] dt < 0.$]

(c) By considering Dulac's criterion with $\phi = 1/(xy)$, show that there are no periodic orbits in Λ if $ab < 0$.

(d) Purely by consideration of the existence of fixed points in Λ and their Poincaré indices, determine those (a, b) for which the possibility of periodic orbits can be excluded.

(e) Combining the results above, sketch the a-b plane showing where periodic orbits in Λ might still be possible.

32C Integrable Systems

Suppose $\psi^s : (x,u) \mapsto (\tilde{x},\tilde{u})$ is a smooth one-parameter group of transformations acting on $\mathbb{R}^2$, with infinitesimal generator

$$V = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}.$$

(a) Define the n^{th} *prolongation* $\mathrm{Pr}^{(n)} V$ of V, and show that

$$\mathrm{Pr}^{(n)} V = V + \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial u^{(i)}},$$

where you should give an explicit formula to determine the η_i recursively in terms of ξ and η.

(b) Find the n^{th} prolongation of each of the following generators:

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = x\frac{\partial}{\partial x}, \qquad V_3 = x^2\frac{\partial}{\partial x}.$$

(c) Given a smooth, real-valued, function $u = u(x)$, the Schwarzian derivative is defined by,

$$S = S[u] := \frac{u_x u_{xxx} - \frac{3}{2}u_{xx}^2}{u_x^2}.$$

Show that,

$$\mathrm{Pr}^{(3)} V_i\,(S) = c_i S,$$

for $i = 1,2,3$ where c_i are real functions which you should determine. What can you deduce about the symmetries of the equations:

(i) $S[u] = 0$,

(ii) $S[u] = 1$,

(iii) $S[u] = \frac{1}{x^2}$?

33B Principles of Quantum Mechanics

Consider the Hamiltonian $H = H_0 + V$, where V is a small perturbation. If $H_0|n\rangle = E_n|n\rangle$, write down an expression for the eigenvalues of H, correct to second order in the perturbation, assuming the energy levels of H_0 are non-degenerate.

In a certain three-state system, H_0 and V take the form

$$H_0 = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \qquad \text{and} \qquad V = V_0 \begin{pmatrix} 0 & \epsilon & \epsilon^2 \\ \epsilon & 0 & 0 \\ \epsilon^2 & 0 & 0 \end{pmatrix},$$

with V_0 and ϵ real, positive constants and $\epsilon \ll 1$.

(a) Consider first the case $E_1 = E_2 \neq E_3$ and $|\epsilon V_0/(E_3 - E_2)| \ll 1$. Use the results of degenerate perturbation theory to obtain the energy eigenvalues correct to order ϵ.

(b) Now consider the different case $E_3 = E_2 \neq E_1$ and $|\epsilon V_0/(E_2 - E_1)| \ll 1$. Use the results of non-degenerate perturbation theory to obtain the energy eigenvalues correct to order ϵ^2. Why is it not necessary to use degenerate perturbation theory in this case?

(c) Obtain the exact energy eigenvalues in case (b), and compare these to your perturbative results by expanding to second order in ϵ.

34B Applications of Quantum Mechanics

A Hamiltonian H is invariant under the discrete translational symmetry of a Bravais lattice Λ. This means that there exists a unitary translation operator $T_{\mathbf{r}}$ such that $[H, T_{\mathbf{r}}] = 0$ for all $\mathbf{r} \in \Lambda$. State and prove *Bloch's theorem* for H.

Consider the two-dimensional Bravais lattice Λ defined by the basis vectors

$$\mathbf{a}_1 = \frac{a}{2}(\sqrt{3}, 1)\,, \quad \mathbf{a}_2 = \frac{a}{2}(\sqrt{3}, -1)\,.$$

Find basis vectors $\mathbf{b_1}$ and $\mathbf{b}_2$ for the reciprocal lattice. Sketch the Brillouin zone. Explain why the Brillouin zone has only two physically distinct corners. Show that the positions of these corners may be taken to be $\mathbf{K} = \frac{1}{3}(2\mathbf{b}_1 + \mathbf{b}_2)$ and $\mathbf{K}' = \frac{1}{3}(\mathbf{b}_1 + 2\mathbf{b}_2)$.

The dynamics of a single electron moving on the lattice Λ is described by a tight-binding model with Hamiltonian

$$H = \sum_{\mathbf{r}\in\Lambda} \Big[E_0 |\mathbf{r}\rangle\langle\mathbf{r}| - \lambda\Big(|\mathbf{r}\rangle\langle\mathbf{r}+\mathbf{a}_1| + |\mathbf{r}\rangle\langle\mathbf{r}+\mathbf{a}_2| + |\mathbf{r}+\mathbf{a}_1\rangle\langle\mathbf{r}| + |\mathbf{r}+\mathbf{a}_2\rangle\langle\mathbf{r}| \Big)\Big]\,,$$

where E_0 and λ are real parameters. What is the energy spectrum as a function of the wave vector $\mathbf{k}$ in the Brillouin zone? How does the energy vary along the boundary of the Brillouin zone between $\mathbf{K}$ and $\mathbf{K}'$? What is the width of the band?

In a real material, each site of the lattice Λ contains an atom with a certain valency. Explain how the conducting properties of the material depend on the valency.

Suppose now that there is a second band, with minimum $E = E_0 + \Delta$. For what values of Δ and the valency is the material an insulator?

35D Statistical Physics

What is meant by the *chemical potential* μ of a thermodynamic system? Derive the Gibbs distribution for a system at temperature T and chemical potential μ (and fixed volume) with variable particle number N.

Consider a non-interacting, two-dimensional gas of N fermionic particles in a region of fixed area, at temperature T and chemical potential μ. Using the Gibbs distribution, find the mean occupation number $n_F(\varepsilon)$ of a one-particle quantum state of energy ε. Show that the density of states $g(\varepsilon)$ is independent of ε and deduce that the mean number of particles between energies ε and $\varepsilon + d\varepsilon$ is very well approximated for $T \ll \varepsilon_F$ by

$$\frac{N}{\varepsilon_F}\frac{d\varepsilon}{e^{(\varepsilon-\varepsilon_F)/T}+1}\,,$$

where ε_F is the Fermi energy. Show that, for T small, the heat capacity of the gas has a power-law dependence on T, and find the power.

36E Electrodynamics

A time-dependent charge distribution $\rho(t, \mathbf{x})$ localised in some region of size a near the origin varies periodically in time with characteristic angular frequency ω. Explain briefly the circumstances under which the *dipole approximation* for the fields sourced by the charge distribution is valid.

Far from the origin, for $r = |\mathbf{x}| \gg a$, the vector potential $\mathbf{A}(t, \mathbf{x})$ sourced by the charge distribution $\rho(t, \mathbf{x})$ is given by the approximate expression

$$\mathbf{A}(t, \mathbf{x}) \;\simeq\; \frac{\mu_0}{4\pi r} \int d^3\mathbf{x}' \, \mathbf{J}\left(t - r/c, \mathbf{x}'\right),$$

where $\mathbf{J}(t, \mathbf{x})$ is the corresponding current density. Show that, in the dipole approximation, the large-distance behaviour of the magnetic field is given by,

$$\mathbf{B}(t, \mathbf{x}) \;\simeq\; -\frac{\mu_0}{4\pi r c} \hat{\mathbf{x}} \times \ddot{\mathbf{p}}\,(t - r/c)\,,$$

where $\mathbf{p}(t)$ is the electric dipole moment of the charge distribution. Assuming that, in the same approximation, the corresponding electric field is given as $\mathbf{E} = -c\hat{\mathbf{x}} \times \mathbf{B}$, evaluate the flux of energy through the surface element of a large sphere of radius R centred at the origin. Hence show that the total power $P(t)$ radiated by the charge distribution is given by

$$P(t) \;=\; \frac{\mu_0}{6\pi c} \left|\ddot{\mathbf{p}}\,(t - R/c)\right|^2 .$$

A particle of charge q and mass m undergoes simple harmonic motion in the x-direction with time period $T = 2\pi/\omega$ and amplitude $\mathcal{A}$ such that

$$\mathbf{x}(t) = \mathcal{A}\,\sin(\omega t)\;\mathbf{i}_x\,. \qquad (\star)$$

Here $\mathbf{i}_x$ is a unit vector in the x-direction. Calculate the total power $P(t)$ radiated through a large sphere centred at the origin in the dipole approximation and determine its time averaged value,

$$\langle P \rangle \;=\; \frac{1}{T} \int_0^T P(t)\, dt\,.$$

For what values of the parameters $\mathcal{A}$ and ω is the dipole approximation valid?

Now suppose that the energy of the particle with trajectory $(\star)$ is given by the usual non-relativistic formula for a harmonic oscillator i.e. $E = m|\dot{\mathbf{x}}|^2/2 + m\omega^2|\mathbf{x}|^2/2$, and that the particle loses energy due to the emission of radiation at a rate corresponding to the time-averaged power $\langle P \rangle$. Work out the half-life of this system (i.e. the time $t_{1/2}$ such that $E(t_{1/2}) = E(0)/2$). Explain why the non-relativistic approximation for the motion of the particle is reliable as long as the dipole approximation is valid.

37D General Relativity

(a) Let $\mathcal{M}$ be a manifold with coordinates x^μ. The commutator of two vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ is defined as

$$[\boldsymbol{V}, \boldsymbol{W}]^\alpha = V^\nu \partial_\nu W^\alpha - W^\nu \partial_\nu V^\alpha \,.$$

(i) Show that $[\boldsymbol{V}, \boldsymbol{W}]$ transforms like a vector field under a change of coordinates from x^μ to $\tilde{x}^\mu$.

(ii) Show that the commutator of any two basis vectors vanishes, i.e.

$$\left[\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right] = 0\,.$$

(iii) Show that if $\boldsymbol{V}$ and $\boldsymbol{W}$ are linear combinations (not necessarily with constant coefficients) of n vector fields $\boldsymbol{Z}_{(a)}$, $a = 1, \ldots, n$ that all commute with one another, then the commutator $[\boldsymbol{V}, \boldsymbol{W}]$ is a linear combination of the same n fields $\boldsymbol{Z}_{(a)}$.

[You may use without proof the following relations which hold for any vector fields $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3$ and any function f:

$$\begin{aligned}
[\boldsymbol{V}_1, \boldsymbol{V}_2] &= -[\boldsymbol{V}_2, \boldsymbol{V}_1]\,, && (1)\\
[\boldsymbol{V}_1, \boldsymbol{V}_2 + \boldsymbol{V}_3] &= [\boldsymbol{V}_1, \boldsymbol{V}_2] + [\boldsymbol{V}_1, \boldsymbol{V}_3]\,, && (2)\\
[\boldsymbol{V}_1, f\boldsymbol{V}_2] &= f\,[\boldsymbol{V}_1, \boldsymbol{V}_2] + \boldsymbol{V}_1(f)\,\boldsymbol{V}_2\,, && (3)
\end{aligned}$$

but you should clearly indicate each time relation (1), (2), or (3) is used.]

(b) Consider the 2-dimensional manifold $\mathbb{R}^2$ with Cartesian coordinates $(x^1, x^2) = (x, y)$ carrying the Euclidean metric $g_{\alpha\beta} = \delta_{\alpha\beta}$.

(i) Express the coordinate basis vectors ∂_r and ∂_θ, where r and θ denote the usual polar coordinates, in terms of their Cartesian counterparts.

(ii) Define the unit vectors

$$\hat{\boldsymbol{r}} = \frac{\partial_r}{||\partial_r||}\,, \qquad \hat{\boldsymbol{\theta}} = \frac{\partial_\theta}{||\partial_\theta||}$$

and show that $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}})$ are *not* a coordinate basis, i.e. there exist no coordinates z^α such that $\hat{\boldsymbol{r}} = \partial/\partial z^1$ and $\hat{\boldsymbol{\theta}} = \partial/\partial z^2$.

38A Fluid Dynamics

For a fluid with kinematic viscosity ν, the steady axisymmetric boundary-layer equations for flow primarily in the z-direction are

$$\begin{aligned} u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} &= \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right), \\ \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} &= 0, \end{aligned}$$

where u is the fluid velocity in the r-direction and w is the fluid velocity in the z-direction. A thin, steady, axisymmetric jet emerges from a point at the origin and flows along the z-axis in a fluid which is at rest far from the z-axis.

(a) Show that the momentum flux

$$M := \int_0^\infty r w^2 dr$$

is independent of the position z along the jet. Deduce that the thickness $\delta(z)$ of the jet increases linearly with z. Determine the scaling dependence on z of the centre-line velocity $W(z)$. Hence show that the jet entrains fluid.

(b) A similarity solution for the streamfunction,

$$\psi(x, y, z) = \nu z g(\eta) \quad \text{with} \quad \eta := r/z,$$

exists if g satisfies the second order differential equation

$$\eta g'' - g' + g g' = 0.$$

Using appropriate boundary and normalisation conditions (which you should state clearly) to solve this equation, show that

$$g(\eta) = \frac{12 M \eta^2}{32\nu^2 + 3M\eta^2}.$$

39A Waves

(a) Derive the wave equation for perturbation pressure for linearised sound waves in a compressible gas.

(b) For a single plane wave show that the perturbation pressure and the velocity are linearly proportional and find the constant of proportionality, i.e. the acoustic impedance.

(c) Gas occupies a tube lying parallel to the x-axis. In the regions $x < 0$ and $x > L$ the gas has uniform density ρ_0 and sound speed c_0. For $0 < x < L$ the temperature of the gas has been adjusted so that it has uniform density ρ_1 and sound speed c_1. A harmonic plane wave with frequency ω and unit amplitude is incident from $x = -\infty$. If T is the (in general complex) amplitude of the wave transmitted into $x > L$, show that

$$|T| = \left(\cos^2 k_1 L + \tfrac{1}{4}\left(\lambda + \lambda^{-1}\right)^2 \sin^2 k_1 L\right)^{-\frac{1}{2}} ,$$

where $\lambda = \rho_1 c_1/\rho_0 c_0$ and $k_1 = \omega/c_1$. Discuss both of the limits $\lambda \ll 1$ and $\lambda \gg 1$.

40C Numerical Analysis

The diffusion equation

$$u_t = u_{xx}, \qquad 0 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

with the initial condition $u(x,0) = \phi(x)$, $0 \leqslant x \leqslant 1$, and boundary conditions $u(0,t) = u(1,t) = 0$, is discretised by $u_m^n \approx u(mh, nk)$ with $k = \Delta t$, $h = \Delta x = 1/(1+M)$. The Courant number is given by $\mu = k/h^2$.

(a) The system is solved numerically by the method

$$u_m^{n+1} = u_m^n + \mu\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right), \qquad m = 1, 2, ..., M, \quad n \geqslant 0\,.$$

Prove directly that $\mu \leqslant 1/2$ implies convergence.

(b) Now consider the method

$$au_m^{n+1} - \tfrac{1}{4}(\mu - c)\left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right) = au_m^n + \tfrac{1}{4}(\mu + c)\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right) ,$$

where a and c are real constants. Using an eigenvalue analysis and carefully justifying each step, determine conditions on μ, a and c for this method to be stable.

[*You may use the notation* $[\beta, \alpha, \beta]$ *for the tridiagonal matrix with* α *along the diagonal, and* β *along the sub- and super-diagonals and use without proof any relevant theorems about such matrices.*]

END OF PAPER