UNIVERSITY OF CAMBRIDGE

## MATHEMATICAL TRIPOS Part IB

Wednesday, 5 June, 2019 9:00 am to 12:00 pm MAT1

## PAPER 2

### Before you begin read these instructions carefully.

*The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.*

*Candidates may obtain credit from attempts on* ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

*Write on* **one side** *of the paper only and begin each answer on a separate sheet.*

*Write legibly; otherwise, you place yourself at a grave disadvantage.*

***At the end of the examination:***

*Tie up your answers in separate bundles labelled* ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

*Attach a completed gold cover sheet to each bundle.*

*You must also complete a green master cover sheet listing all the questions you have attempted.*

***Every cover sheet must bear your examination number and desk number.***

***STATIONERY REQUIREMENTS***
*Gold cover sheet*
*Green master cover sheet*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

# SECTION I

## 1F Linear Algebra

If $U$ and $W$ are finite-dimensional subspaces of a vector space $V$, prove that

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)\,.$$

Let

$$\begin{aligned} U &= \{\mathbf{x} \in \mathbb{R}^4 \,|\, x_1 = 7x_3 + 8x_4,\ x_2 + 5x_3 + 6x_4 = 0\}\,, \\ W &= \{\mathbf{x} \in \mathbb{R}^4 \,|\, x_1 + 2x_2 + 3x_3 = 0,\ x_4 = 0\}\,. \end{aligned}$$

Show that $U+W$ is 3-dimensional and find a linear map $\ell : \mathbb{R}^4 \to \mathbb{R}$ such that

$$U + W = \{\mathbf{x} \in \mathbb{R}^4 \,|\, \ell(\mathbf{x}) = 0\}\,.$$

## 2G Groups, Rings and Modules

Let $R$ be an integral domain. A module $M$ over $R$ is *torsion-free* if, for any $r \in R$ and $m \in M$, $rm = 0$ only if $r = 0$ or $m = 0$.

Let $M$ be a module over $R$. Prove that there is a quotient

$$q : M \to M_0$$

with $M_0$ torsion-free and with the following property: whenever $N$ is a torsion-free module and $f : M \to N$ is a homomorphism of modules, there is a homomorphism $f_0 : M_0 \to N$ such that $f = f_0 \circ q$.

## 3E Analysis II

Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (x^{1/3} + y^2, y^5)$$

where $x^{1/3}$ denotes the unique real cube root of $x \in \mathbb{R}$.

(a) At what points is $f$ continuously differentiable? Calculate its derivative there.

(b) Show that $f$ has a local differentiable inverse near any $(x,y)$ with $xy \neq 0$.

You should justify your answers, stating accurately any results that you require.

**4G Metric and Topological Spaces**

(a) Let $f : X \to Y$ be a continuous surjection of topological spaces. Prove that, if $X$ is connected, then $Y$ is also connected.

(b) Let $g : [0,1] \to [0,1]$ be a continuous map. Deduce from part (a) that, for every $y$ between $g(0)$ and $g(1)$, there is $x \in [0,1]$ such that $g(x) = y$. [You may not assume the Intermediate Value Theorem, but you may use the fact that suprema exist in $\mathbb{R}$.]

**5B Methods**

Let $r$, $\theta$, $\phi$ be spherical polar coordinates, and let $P_n$ denote the $n$th Legendre polynomial. Write down the most general solution for $r > 0$ of Laplace's equation $\nabla^2\Phi = 0$ that takes the form $\Phi(r,\theta,\phi) = f(r)P_n(\cos\theta)$.

Solve Laplace's equation in the spherical shell $1 \leqslant r \leqslant 2$ subject to the boundary conditions

$$\begin{aligned} \Phi &= 3\cos 2\theta \quad \text{at} \quad r = 1\,, \\ \Phi &= 0 \qquad\qquad \text{at} \quad r = 2\,. \end{aligned}$$

[*The first three Legendre polynomials are*

$$P_0(x) = 1, \quad P_1(x) = x \quad \textit{and} \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}\,.]$$

**6A Electromagnetism**

Write down the solution for the scalar potential $\varphi(\mathbf{x})$ that satisfies

$$\nabla^2 \varphi = -\frac{1}{\varepsilon_0} \rho \,,$$

with $\varphi(\mathbf{x}) \to 0$ as $r = |\mathbf{x}| \to \infty$. You may assume that the charge distribution $\rho(\mathbf{x})$ vanishes for $r > R$, for some constant $R$. In an expansion of $\varphi(\mathbf{x})$ for $r \gg R$, show that the terms of order $1/r$ and $1/r^2$ can be expressed in terms of the total charge $Q$ and the electric dipole moment $\mathbf{p}$, which you should define.

Write down the analogous solution for the vector potential $\mathbf{A}(\mathbf{x})$ that satisfies

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \,,$$

with $\mathbf{A}(\mathbf{x}) \to \mathbf{0}$ as $r \to \infty$. You may assume that the current $\mathbf{J}(\mathbf{x})$ vanishes for $r > R$ and that it obeys $\nabla \cdot \mathbf{J} = 0$ everywhere. In an expansion of $\mathbf{A}(\mathbf{x})$ for $r \gg R$, show that the term of order $1/r$ vanishes.

[*Hint:* $\frac{\partial}{\partial x_j}(x_i J_j) = J_i + x_i \frac{\partial J_j}{\partial x_j}$ .]

**7C Fluid Dynamics**

Consider the steady flow

$$u_x = \sin x \cos y \,, \qquad u_y = -\cos x \sin y \,, \qquad u_z = 0 \,,$$

where $(x, y, z)$ are Cartesian coordinates. Show that $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and determine the streamfunction. Calculate the vorticity and verify that the vorticity equation is satisfied in the absence of viscosity. Sketch the streamlines in the region $0 < x < 2\pi$, $0 < y < 2\pi$.

**8H Statistics**

Suppose that $X_1, \ldots, X_n$ are i.i.d. coin tosses with probability $\theta$ of obtaining a head.

(a) Compute the posterior distribution of $\theta$ given the observations $X_1, \ldots, X_n$ in the case of a uniform prior on $[0, 1]$.

(b) Give the definition of the *quadratic error loss function.*

(c) Determine the value $\widehat{\theta}$ of $\theta$ which minimizes the quadratic error loss function. Justify your answer. Calculate $\mathbb{E}[\widehat{\theta}]$.

[*You may use that the* $\beta(a, b)$, $a, b > 0$, *distribution has density function on* $[0, 1]$ *given by*

$$c_{a,b}\, x^{a-1}(1-x)^{b-1}$$

*where* $c_{a,b}$ *is a normalizing constant. You may also use without proof that the mean of a* $\beta(a, b)$ *random variable is* $a/(a+b)$.]

**9H Optimisation**

State the Lagrange sufficiency theorem.

Find the maximum of $\log(xyz)$ over $x, y, z > 0$ subject to the constraint

$$x^2 + y^2 + z^2 = 1$$

using Lagrange multipliers. Carefully justify why your solution is in fact the maximum.

Find the maximum of $\log(xyz)$ over $x, y, z > 0$ subject to the constraint

$$x^2 + y^2 + z^2 \leqslant 1.$$

## SECTION II

**10F Linear Algebra**

Let $A$ and $B$ be $n \times n$ matrices over $\mathbb{C}$.

(a) Assuming that $A$ is invertible, show that $AB$ and $BA$ have the same characteristic polynomial.

(b) By considering the matrices $A - sI$, show that $AB$ and $BA$ have the same characteristic polynomial even when $A$ is singular.

(c) Give an example to show that the minimal polynomials $m_{AB}(t)$ and $m_{BA}(t)$ of $AB$ and $BA$ may be different.

(d) Show that $m_{AB}(t)$ and $m_{BA}(t)$ differ at most by a factor of $t$. Stating carefully any results which you use, deduce that if $AB$ is diagonalisable then so is $(BA)^2$.

**11G Groups, Rings and Modules**

(a) Let $k$ be a field and let $f(X)$ be an irreducible polynomial of degree $d > 0$ over $k$. Prove that there exists a field $F$ containing $k$ as a subfield such that

$$f(X) = (X - \alpha)g(X)\,,$$

where $\alpha \in F$ and $g(X) \in F[X]$. State carefully any results that you use.

(b) Let $k$ be a field and let $f(X)$ be a monic polynomial of degree $d > 0$ over $k$, which is not necessarily irreducible. Prove that there exists a field $F$ containing $k$ as a subfield such that

$$f(X) = \prod_{i=1}^{d}(X - \alpha_i)\,,$$

where $\alpha_i \in F$.

(c) Let $k = \mathbb{Z}/(p)$ for $p$ a prime, and let $f(X) = X^{p^n} - X$ for $n \geqslant 1$ an integer. For $F$ as in part (b), let $K$ be the set of roots of $f(X)$ in $F$. Prove that $K$ is a field.

**12E Analysis II**

(a) (i) Define what it means for two norms on a vector space to be *Lipschitz equivalent*.

(ii) Show that any two norms on a finite-dimensional vector space are Lipschitz equivalent.

(iii) Show that if two norms $||\cdot||$, $||\cdot||'$ on a vector space $V$ are Lipschitz equivalent then the following holds: for any sequence $(v_n)$ in $V$, $(v_n)$ is Cauchy with respect to $||\cdot||$ if and only if it is Cauchy with respect to $||\cdot||'$.

(b) Let $V$ be the vector space of real sequences $x = (x_i)$ such that $\sum |x_i| < \infty$. Let

$$||x||_\infty = \sup\{|x_i| : i \in \mathbb{N}\},$$

and for $1 \leqslant p < \infty$, let

$$||x||_p = \left(\sum |x_i|^p\right)^{1/p}.$$

You may assume that $||\cdot||_\infty$ and $||\cdot||_p$ are well-defined norms on $V$.

(i) Show that $||\cdot||_p$ is not Lipschitz equivalent to $||\cdot||_\infty$ for any $1 \leqslant p < \infty$.

(ii) Are there any $p, q$ with $1 \leqslant p < q < \infty$ such that $||\cdot||_p$ and $||\cdot||_q$ are Lipschitz equivalent? Justify your answer.

**13D Complex Analysis or Complex Methods**

Let $C_1$ and $C_2$ be smooth curves in the complex plane, intersecting at some point $p$. Show that if the map $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable, then it preserves the angle between $C_1$ and $C_2$ at $p$, provided $f'(p) \neq 0$. Give an example that illustrates why the condition $f'(p) \neq 0$ is important.

Show that $f(z) = z + 1/z$ is a one-to-one conformal map on each of the two regions $|z| > 1$ and $0 < |z| < 1$, and find the image of each region.

Hence construct a one-to-one conformal map from the unit disc to the complex plane with the intervals $(-\infty, -1/2]$ and $[1/2, \infty)$ removed.

**14E Geometry**

Define a *smooth embedded surface* in $\mathbb{R}^3$. Sketch the surface $C$ given by

$$\left(\sqrt{2x^2+2y^2}-4\right)^2+2z^2=2$$

and find a smooth parametrisation for it. Use this to calculate the Gaussian curvature of $C$ at every point.

Hence or otherwise, determine which points of the embedded surface

$$\left(\sqrt{x^2+2xz+z^2+2y^2}-4\right)^2+(z-x)^2=2$$

have Gaussian curvature zero. [*Hint: consider a transformation of* $\mathbb{R}^3$.]

[*You should carefully state any result that you use.*]

**15A Variational Principles**

Write down the Euler–Lagrange (EL) equations for a functional

$$\int_a^b f(\,u,\,w,\,u',\,w',\,x\,)\,dx\,,$$

where $u(x)$ and $w(x)$ each take specified values at $x=a$ and $x=b$. Show that the EL equations imply that

$$\kappa \;=\; f \;-\; u'\frac{\partial f}{\partial u'} \;-\; w'\frac{\partial f}{\partial w'}$$

is independent of $x$ provided $f$ satisfies a certain condition, to be specified. State conditions under which there exist additional first integrals of the EL equations.

Consider

$$f \;=\; \left(1-\frac{m}{u}\right)w'^{\,2} \;-\; \left(1-\frac{m}{u}\right)^{-1}u'^{\,2}$$

where $m$ is a positive constant. Show that solutions of the EL equations satisfy

$$u'^{\,2} \;=\; \lambda^2 \;+\; \kappa\left(1-\frac{m}{u}\right),$$

for some constant $\lambda$. Assuming that $\kappa=-\lambda^2$, find $dw/du$ and hence determine the most general solution for $w$ as a function of $u$ subject to the conditions $u>m$ and $w\to-\infty$ as $u\to\infty$. Show that, for any such solution, $w\to\infty$ as $u\to m$.

[*Hint:*

$$\frac{d}{dz}\left\{\,\log\left(\frac{z^{1/2}-1}{z^{1/2}+1}\right)\right\} \;=\; \frac{1}{z^{1/2}\,(z-1)}\,.\quad ]$$

**16D Methods**

For $n = 0, 1, 2, \ldots$, the degree $n$ polynomial $C_n^\alpha(x)$ satisfies the differential equation

$$(1-x^2)y'' - (2\alpha+1)xy' + n(n+2\alpha)y = 0$$

where $\alpha$ is a real, positive parameter. Show that, when $m \neq n$,

$$\int_a^b C_m^\alpha(x)\, C_n^\alpha(x)\, w(x)\, dx = 0$$

for a weight function $w(x)$ and values $a < b$ that you should determine.

Suppose that the roots of $C_n^\alpha(x)$ that lie inside the domain $(a,b)$ are $\{x_1, x_2, \ldots, x_k\}$, with $k \leqslant n$. By considering the integral

$$\int_a^b C_n^\alpha(x) \prod_{i=1}^k (x - x_i)\, w(x)\, dx\,,$$

show that in fact all $n$ roots of $C_n^\alpha(x)$ lie in $(a,b)$.

**17B Quantum Mechanics**

Let $x$, $y$, $z$ be Cartesian coordinates in $\mathbb{R}^3$. The angular momentum operators satisfy the commutation relation

$$[L_x, L_y] = i\hbar L_z$$

and its cyclic permutations. Define the *total angular momentum operator* $\mathbf{L}^2$ and show that $[L_z, \mathbf{L}^2] = 0$. Write down the explicit form of $L_z$.

Show that a function of the form $(x+iy)^m z^n f(r)$, where $r^2 = x^2 + y^2 + z^2$, is an eigenfunction of $L_z$ and find the eigenvalue. State the analogous result for $(x-iy)^m z^n f(r)$.

There is an energy level for a particle in a certain spherically symmetric potential well that is 6-fold degenerate. A basis for the (unnormalized) energy eigenstates is of the form

$$(x^2-1)f(r),\ (y^2-1)f(r),\ (z^2-1)f(r),\ xyf(r),\ xzf(r),\ yzf(r)\,.$$

Find a new basis that consists of simultaneous eigenstates of $L_z$ and $\mathbf{L}^2$ and identify their eigenvalues.

[*You may quote the range of $L_z$ eigenvalues associated with a particular eigenvalue of* $\mathbf{L}^2$.]

**18A Electromagnetism**

Consider a conductor in the shape of a closed curve $C$ moving in the presence of a magnetic field $\mathbf{B}$. State Faraday's Law of Induction, defining any quantities that you introduce.

Suppose $C$ is a square horizontal loop that is allowed to move only vertically. The location of the loop is specified by a coordinate $z$, measured vertically upwards, and the edges of the loop are defined by $x = \pm a$, $-a \leqslant y \leqslant a$ and $y = \pm a$, $-a \leqslant x \leqslant a$. If the magnetic field is

$$\mathbf{B} = b\,(\,x,\, y,\, -2z\,)\,,$$

where $b$ is a constant, find the induced current $I$, given that the total resistance of the loop is $R$.

Calculate the resulting electromagnetic force on the edge of the loop $x = a$, and show that this force acts at an angle $\tan^{-1}(2z/a)$ to the vertical. Find the total electromagnetic force on the loop and comment on its direction.

Now suppose that the loop has mass $m$ and that gravity is the only other force acting on it. Show that it is possible for the loop to fall with a constant downward velocity $Rmg/(8ba^2)^2$.

**19C Numerical Analysis**

Define the *linear least squares problem* for the equation

$$A\mathbf{x} = \mathbf{b}\,,$$

where $A$ is a given $m \times n$ matrix with $m > n$, $\mathbf{b} \in \mathbb{R}^m$ is a given vector and $\mathbf{x} \in \mathbb{R}^n$ is an unknown vector.

Explain how the linear least squares problem can be solved by obtaining a $QR$ factorization of the matrix $A$, where $Q$ is an orthogonal $m \times m$ matrix and $R$ is an upper-triangular $m \times n$ matrix in standard form.

Use the Gram–Schmidt method to obtain a $QR$ factorization of the matrix

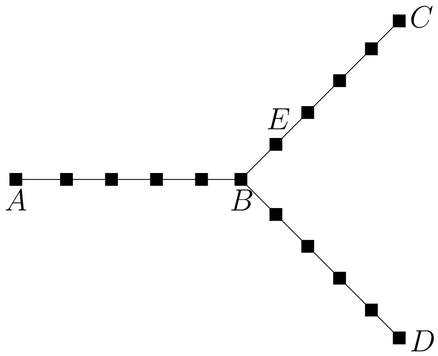
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and use it to solve the linear least squares problem in the case

$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}\,.$$

**20H Markov Chains**

Fix $n \geqslant 1$ and let $G$ be the graph consisting of a copy of $\{0, \ldots, n\}$ joining vertices $A$ and $B$, a copy of $\{0, \ldots, n\}$ joining vertices $B$ and $C$, and a copy of $\{0, \ldots, n\}$ joining vertices $B$ and $D$. Let $E$ be the vertex adjacent to $B$ on the segment from $B$ to $C$. Shown below is an illustration of $G$ in the case $n = 5$. The vertices are solid squares and edges are indicated by straight lines.




Let $(X_k)$ be a simple random walk on $G$. In other words, in each time step, $X$ moves to one of its neighbours with equal probability. Assume that $X_0 = A$.

(a) Compute the expected amount of time for $X$ to hit $B$.

(b) Compute the expected amount of time for $X$ to hit $E$. [*Hint: first show that the expected amount of time $x$ for $X$ to go from $B$ to $E$ satisfies $x = \frac{1}{3} + \frac{2}{3}(L + x)$ where $L$ is the expected return time of $X$ to $B$ when starting from $B$.*]

(c) Compute the expected amount of time for $X$ to hit $C$. [*Hint: for each $i$, let $v_i$ be the vertex which is $i$ places to the right of $B$ on the segment from $B$ to $C$. Derive an equation for the expected amount of time $x_i$ for $X$ to go from $v_i$ to $v_{i+1}$.*]

Justify all of your answers.

## END OF PAPER