UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday, 4 June, 2019 9:00 am to 12:00 pm MAT1

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Define a *basis* of a vector space V.

If V has a finite basis $\mathcal{B}$, show using only the definition that any other basis $\mathcal{B}'$ has the same cardinality as $\mathcal{B}$.

2F Complex Analysis or Complex Methods

What is the *Laurent series* for a function f defined in an annulus A? Find the Laurent series for $f(z) = \frac{10}{(z+2)(z^2+1)}$ on the annuli

$$A_1 = \{z \in \mathbb{C} \mid 0 < |z| < 1\} \quad \text{and}$$
$$A_2 = \{z \in \mathbb{C} \mid 1 < |z| < 2\}\,.$$

3E Geometry

Describe the Poincaré disc model D for the hyperbolic plane by giving the appropriate Riemannian metric.

Calculate the distance between two points $z_1, z_2 \in D$. You should carefully state any results about isometries of D that you use.

4A Variational Principles

A function $\phi = xy - yz$ is defined on the surface $x^2 + 2y^2 + z^2 = 1$. Find the location (x, y, z) of every stationary point of this function.

5C Fluid Dynamics

A viscous fluid flows steadily down a plane that is inclined at an angle α to the horizontal. The fluid layer is of uniform thickness and has a free upper surface. Determine the velocity profile in the direction perpendicular to the plane and also the volume flux (per unit width), in terms of the gravitational acceleration g, the angle α, the kinematic viscosity ν and the thickness h of the fluid layer.

Show that the volume flux is reduced if the free upper surface is replaced by a stationary plane boundary, and give a physical explanation for this.

6C Numerical Analysis

Let $[a, b]$ be the smallest interval that contains the $n + 1$ distinct real numbers $x_0, x_1, \ldots, x_n$, and let f be a continuous function on that interval.

Define the *divided difference* $f[x_0, x_1, \ldots, x_m]$ of degree $m \leqslant n$.

Prove that the polynomial of degree n that interpolates the function f at the points $x_0, x_1, \ldots, x_n$ is equal to the Newton polynomial

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \ldots, x_n] \prod_{i=0}^{n-1} (x - x_i)\,.$$

Prove the recursive formula

$$f[x_0, x_1, \ldots, x_m] = \frac{f[x_1, x_2, \ldots, x_m] - f[x_0, x_1, \ldots, x_{m-1}]}{x_m - x_0}$$

for $1 \leqslant m \leqslant n$.

7H Statistics

Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$ random variables.

(a) Compute the MLEs $\widehat{\mu}$, $\widehat{\sigma}^2$ for the unknown parameters μ, σ^2.

(b) Give the definition of an *unbiased estimator*. Determine whether $\widehat{\mu}$, $\widehat{\sigma}^2$ are unbiased estimators for μ, σ^2.

8H Optimisation

Suppose that f is an infinitely differentiable function on $\mathbb{R}$. Assume that there exist constants $0 < C_1, C_2 < \infty$ so that $|f''(x)| \geqslant C_1$ and $|f'''(x)| \leqslant C_2$ for all $x \in \mathbb{R}$. Fix $x_0 \in \mathbb{R}$ and for each $n \in \mathbb{N}$ set

$$x_n = x_{n-1} - \frac{f'(x_{n-1})}{f''(x_{n-1})}\,.$$

Let x^* be the unique value of x where f attains its minimum. Prove that

$$|x^* - x_{n+1}| \leqslant \frac{C_2}{2C_1}|x^* - x_n|^2 \quad \text{for all } n \in \mathbb{N}.$$

[*Hint: Express $f'(x^*)$ in terms of the Taylor series for f' at x_n using the Lagrange form of the remainder: $f'(x^*) = f'(x_n) + f''(x_n)(x^* - x_n) + \frac{1}{2}f'''(y_n)(x^* - x_n)^2$ where y_n is between x_n and x^*.*]

SECTION II

9F Linear Algebra

What is the *adjugate* $\operatorname{adj}(A)$ of an $n \times n$ matrix A? How is it related to $\det(A)$?

(a) Define matrices $B_0, B_1, \ldots, B_{n-1}$ by

$$\operatorname{adj}(tI - A) = \sum_{i=0}^{n-1} B_i t^{n-1-i}$$

and scalars $c_0, c_1, \ldots, c_n$ by

$$\det(tI - A) = \sum_{j=0}^{n} c_j t^{n-j} .$$

Find a recursion for the matrices B_i in terms of A and the c_j's.

(b) By considering the partial derivatives of the multivariable polynomial

$$p(t_1, t_2, \ldots, t_n) = \det\left(\begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix} - A\right),$$

show that

$$\frac{d}{dt}(\det(tI - A)) = \operatorname{Tr}(\operatorname{adj}(tI - A)) .$$

(c) Hence show that the c_j's may be expressed in terms of $\operatorname{Tr}(A), \operatorname{Tr}(A^2), \ldots, \operatorname{Tr}(A^n)$.

10G Groups, Rings and Modules

(a) Let G be a group of order p^4, for p a prime. Prove that G is not simple.

(b) State Sylow's theorems.

(c) Let G be a group of order p^2q^2, where p, q are distinct odd primes. Prove that G is not simple.

11E Analysis II

Let $A \subset \mathbb{R}^n$ be an open subset. State what it means for a function $f : A \to \mathbb{R}^m$ to be *differentiable* at a point $p \in A$, and define its derivative $Df(p)$.

State and prove the chain rule for the derivative of $g \circ f$, where $g : \mathbb{R}^m \to \mathbb{R}^r$ is a differentiable function.

Let $M = M_n(\mathbb{R})$ be the vector space of $n \times n$ real-valued matrices, and $V \subset M$ the open subset consisting of all invertible ones. Let $f : V \to V$ be given by $f(A) = A^{-1}$.

(a) Show that f is differentiable at the identity matrix, and calculate its derivative.

(b) For $C \in V$, let $l_C, r_C : M \to M$ be given by $l_C(A) = CA$ and $r_C(A) = AC$. Show that $r_C \circ f \circ l_C = f$ on V. Hence or otherwise, show that f is differentiable at any point of V, and calculate $Df(C)(h)$ for $h \in M$.

12G Metric and Topological Spaces

Consider the set of sequences of integers

$$X = \{(x_1, x_2, \ldots) \mid x_n \in \mathbb{Z} \text{ for all } n\}.$$

Define

$$n_{\min}((x_n), (y_n)) = \begin{cases} \infty & x_n = y_n \text{ for all } n \\ \min\{n \mid x_n \neq y_n\} & \text{otherwise} \end{cases}$$

for two sequences $(x_n), (y_n) \in X$. Let

$$d((x_n), (y_n)) = 2^{-n_{\min}((x_n),(y_n))}$$

where, as usual, we adopt the convention that $2^{-\infty} = 0$.

(a) Prove that d defines a metric on X.

(b) What does it mean for a metric space to be *complete*? Prove that (X, d) is complete.

(c) Is (X, d) path connected? Justify your answer.

13F Complex Analysis or Complex Methods

State and prove Jordan's lemma.

What is the *residue* of a function f at an isolated singularity a? If $f(z) = \frac{g(z)}{(z-a)^k}$ with k a positive integer, g analytic, and $g(a) \neq 0$, derive a formula for the residue of f at a in terms of derivatives of g.

Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(1+x^2)^2}\, dx\,.$$

14B Methods

The Bessel functions $J_n(r)$ $(n \geqslant 0)$ can be defined by the expansion

$$e^{ir\cos\theta} = J_0(r) + 2\sum_{n=1}^{\infty} i^n J_n(r)\cos n\theta\,. \qquad (*)$$

By using Cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$, or otherwise, show that

$$(\nabla^2 + 1)e^{ir\cos\theta} = 0\,.$$

Deduce that $J_n(r)$ satisfies Bessel's equation

$$\left(r^2\frac{d^2}{dr^2} + r\frac{d}{dr} - (n^2 - r^2)\right) J_n(r) = 0\,.$$

By expanding the left-hand side of $(*)$ up to cubic order in r, derive the series expansions of $J_0(r)$, $J_1(r)$, $J_2(r)$ and $J_3(r)$ up to this order.

15B Quantum Mechanics

Starting from the time-dependent Schrödinger equation, show that a stationary state $\psi(x)$ of a particle of mass m in a harmonic oscillator potential in one dimension with frequency ω satisfies

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi\,.$$

Find a rescaling of variables that leads to the simplified equation

$$-\frac{d^2\psi}{dy^2}+y^2\psi=\varepsilon\psi\,.$$

Setting $\psi=f(y)e^{-\frac{1}{2}y^2}$, find the equation satisfied by $f(y)$.

Assume now that f is a polynomial

$$f(y)=y^N+a_{N-1}\,y^{N-1}+a_{N-2}\,y^{N-2}+\ldots+a_0\,.$$

Determine the value of ε and deduce the corresponding energy level E of the harmonic oscillator. Show that if N is even then the stationary state $\psi(x)$ has even parity.

16A Electromagnetism

Let $\mathbf{E}(\mathbf{x})$ be the electric field and $\varphi(\mathbf{x})$ the scalar potential due to a static charge density $\rho(\mathbf{x})$, with all quantities vanishing as $r=|\mathbf{x}|$ becomes large. The electrostatic energy of the configuration is given by

$$U\;=\;\frac{\varepsilon_0}{2}\int|\mathbf{E}|^2\,dV\;=\;\frac{1}{2}\int\rho\,\varphi\,dV\,,\qquad(*)$$

with the integrals taken over all space. Verify that these integral expressions agree.

Suppose that a total charge Q is distributed uniformly in the region $a\leqslant r\leqslant b$ and that $\rho=0$ otherwise. Use the integral form of Gauss's Law to determine $\mathbf{E}(\mathbf{x})$ at all points in space and, without further calculation, sketch graphs to indicate how $|\mathbf{E}|$ and φ depend on position.

Consider the limit $b\to a$ with Q fixed. Comment on the continuity of $\mathbf{E}$ and φ. Verify directly from each of the integrals in $(*)$ that $U=Q\,\varphi(a)/2$ in this limit.

Now consider a small change δQ in the total charge Q. Show that the first-order change in the energy is $\delta U=\delta Q\,\varphi(a)$ and interpret this result.

17C Fluid Dynamics

Explain why the irrotational flow of an incompressible fluid can be expressed in terms of a velocity potential ϕ that satisfies Laplace's equation.

The axis of a stationary cylinder of radius a coincides with the z-axis of a Cartesian coordinate system (x, y, z) with unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. A fluid of density ρ flows steadily past the cylinder such that the velocity field $\mathbf{u}$ is independent of z and has no component in the z-direction. The flow is irrotational but there is a constant non-zero circulation

$$\oint \mathbf{u} \cdot d\mathbf{r} = \kappa$$

around every closed curve that encloses the cylinder once in a positive sense. Far from the cylinder, the velocity field tends towards the uniform flow $\mathbf{u} = U\,\mathbf{e}_x$, where U is a constant.

State the boundary conditions on the velocity potential, in terms of polar coordinates (r, θ) in the (x, y)-plane. Explain why the velocity potential is not required to be a single-valued function of position. Hence obtain the appropriate solution $\phi(r, \theta)$, in terms of a, U and κ.

Neglecting gravity, show that the net force on the cylinder, per unit length in the z-direction, is

$$-\rho\kappa U\,\mathbf{e}_y\,.$$

Determine the number and location of stagnation points in the flow as a function of the dimensionless parameter

$$\lambda = \frac{\kappa}{4\pi U a}\,.$$

18C Numerical Analysis

(a) An s-step method for solving the ordinary differential equation

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y})$$

is given by

$$\sum_{l=0}^{s} \rho_l \, \mathbf{y}_{n+l} = h \sum_{l=0}^{s} \sigma_l \, \mathbf{f}(t_{n+l}, \mathbf{y}_{n+l}), \qquad n = 0, 1, \ldots,$$

where ρ_l and σ_l $(l = 0, 1, \ldots, s)$ are constant coefficients, with $\rho_s = 1$, and h is the time-step. Prove that the method is of order $p \geqslant 1$ if and only if

$$\rho(e^z) - z\sigma(e^z) = O(z^{p+1})$$

as $z \to 0$, where

$$\rho(w) = \sum_{l=0}^{s} \rho_l w^l, \qquad \sigma(w) = \sum_{l=0}^{s} \sigma_l w^l.$$

(b) Show that the Adams–Moulton method

$$\mathbf{y}_{n+2} = \mathbf{y}_{n+1} + \frac{h}{12}\Big(5\,\mathbf{f}(t_{n+2}, \mathbf{y}_{n+2}) + 8\,\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) - \mathbf{f}(t_n, \mathbf{y}_n)\Big)$$

is of third order and convergent.

[*You may assume the Dahlquist equivalence theorem if you state it clearly.*]

19H Statistics

State and prove the Neyman–Pearson lemma.

Suppose that $X_1, \ldots, X_n$ are i.i.d. $\exp(\lambda)$ random variables where λ is an unknown parameter. We wish to test the hypothesis $H_0 : \lambda = \lambda_0$ against the hypothesis $H_1 : \lambda = \lambda_1$ where $\lambda_1 < \lambda_0$.

(a) Find the critical region of the likelihood ratio test of size α in terms of the sample mean $\overline{X}$.

(b) Define the *power function* of a hypothesis test and identify the power function in the setting described above in terms of the $\Gamma(n, \lambda)$ probability distribution function. [You may use without proof that $X_1 + \cdots + X_n$ is distributed as a $\Gamma(n, \lambda)$ random variable.]

(c) Define what it means for a hypothesis test to be *uniformly most powerful.* Determine whether the likelihood ratio test considered above is uniformly most powerful for testing $H_0 : \lambda = \lambda_0$ against $\widetilde{H}_1 : \lambda < \lambda_0$.

20H Markov Chains

Let P be a transition matrix for a Markov chain (X_n) on a state space with N elements with $N < \infty$. Assume that the Markov chain is aperiodic and irreducible and let π be its unique invariant distribution. Assume that $X_0 \sim \pi$.

(a) Let $P^*(x,y) = \mathbb{P}[X_0 = y \,|\, X_1 = x]$. Show that $P^*(x,y) = \pi(y)P(y,x)/\pi(x)$.

(b) Let $T = \min\{n \geqslant 1 : X_n = X_0\}$. Compute $\mathbb{E}[T]$ in terms of an explicit function of N.

(c) Suppose that a cop and a robber start from a common state chosen from π. The robber then takes one step according to P^* and stops. The cop then moves according to P independently of the robber until the cop catches the robber (i.e., the cop visits the state occupied by the robber). Compute the expected amount of time for the cop to catch the robber.

END OF PAPER