UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Wednesday, 5 June, 2019 1:30 pm to 4:30 pm MAT0

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A****,* ***B****,* ***C****,* ***D****,* ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

Find all solutions to the simultaneous congruences

$$4x \equiv 1 \pmod{21} \qquad \text{and} \qquad 2x \equiv 5 \pmod{45}.$$

2E Numbers and Sets

Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{(2n-1)!}$$

converge. Determine in each case whether the limit is a rational number. Justify your answers.

3A Dynamics and Relativity

A rocket of mass $m(t)$ moving at speed $v(t)$ and ejecting fuel behind it at a constant speed u relative to the rocket, is subject to an external force F. Considering a small time interval δt, derive the rocket equation

$$m\frac{dv}{dt} + u\frac{dm}{dt} = F.$$

In deep space where $F = 0$, how much faster does the rocket go if it burns half of its mass in fuel?

4A Dynamics and Relativity

Galileo releases a cannonball of mass m from the top of the leaning tower of Pisa, a vertical height h above the ground. Ignoring the rotation of the Earth but assuming that the cannonball experiences a quadratic drag force whose magnitude is γv^2 (where v is the speed of the cannonball), find the time for it to hit the ground in terms of h, m, γ and g, the acceleration due to gravity. [You may assume that g is constant.]

SECTION II

5E Numbers and Sets

(a) State and prove Fermat's theorem. Use it to compute $3^{803} \pmod{17}$.

(b) The *Fibonacci numbers* $F_0, F_1, F_2, \ldots$ are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 2$. Prove by induction that for all $n \geqslant 1$ we have

$$F_{2n} = F_n(F_{n-1} + F_{n+1}) \qquad \text{and} \qquad F_{2n+1} = F_n^2 + F_{n+1}^2.$$

(c) Let $m \geqslant 1$ and let p be an odd prime dividing F_m. Which of the following statements are true, and which can be false? Justify your answers.

(i) If m is odd then $p \equiv 1 \pmod 4$.

(ii) If m is even then $p \equiv 3 \pmod 4$.

6E Numbers and Sets

State the inclusion-exclusion principle.

Let $n \geqslant 2$ be an integer. Let $X = \{0, 1, 2, \ldots, n-1\}$ and

$$Y = \{(a,b) \in X^2 \,|\, \gcd(a,b,n) = 1\}$$

where $\gcd(x_1, \ldots, x_k)$ is the largest number dividing all of $x_1, \ldots, x_k$. Let R be the relation on Y where $(a,b)R(c,d)$ if $ad - bc \equiv 0 \pmod n$.

(a) Show that

$$|Y| = n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right)$$

where the product is over all primes p dividing n.

(b) Show that if $\gcd(a,b,n) = 1$ then there exist integers r, s, t with $ra + sb + tn = 1$.

(c) Show that if $(a,b)R(c,d)$ then there exists an integer λ with $\lambda a \equiv c \pmod n$ and $\lambda b \equiv d \pmod n$. [*Hint: Consider $\lambda = rc + sd$, where r, s are as in part (b).*] Deduce that R is an equivalence relation.

(d) What is the size of the equivalence class containing $(1,1)$? Show that all equivalence classes have the same size, and deduce that the number of equivalence classes is

$$n \prod_{p|n} \left(1 + \frac{1}{p}\right).$$

7E Numbers and Sets

(a) Let $f : X \to Y$ be a function. Show that the following statements are equivalent.

(i) f is injective.

(ii) For every subset $A \subset X$ we have $f^{-1}(f(A)) = A$.

(iii) For every pair of subsets $A, B \subset X$ we have $f(A \cap B) = f(A) \cap f(B)$.

(b) Let $f : X \to X$ be an injection. Show that $X = A \cup B$ for some subsets $A, B \subset X$ such that

$$\bigcap_{n=1}^{\infty} f^n(A) = \emptyset \qquad \text{and} \qquad f(B) = B.$$

[*Here f^n denotes the n-fold composite of f with itself.*]

8E Numbers and Sets

(a) What is a *countable set*? Let X, A, B be sets with A, B countable. Show that if $f : X \to A \times B$ is an injection then X is countable. Deduce that $\mathbb{Z}$ and $\mathbb{Q}$ are countable. Show too that a countable union of countable sets is countable.

(b) Show that, in the plane, any collection of pairwise disjoint circles with rational radius is countable.

(c) A *lollipop* is any subset of the plane obtained by translating, rotating and scaling (by any factor $\lambda > 0$) the set

$$\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\} \cup \{(0, y) \in \mathbb{R}^2 | -3 \leqslant y \leqslant -1\}.$$

What happens if in part (b) we replace 'circles with rational radius' by 'lollipops'?

9A Dynamics and Relativity

In an alien invasion, a flying saucer hovers at a fixed point S, a height l far above the White House, which is at point W. A wrecking ball of mass m is attached to one end of a light inextensible rod, also of length l. The other end of the rod is attached to the flying saucer. The wrecking ball is initially at rest at point B, and the angle WSB is θ_0. At W, the acceleration due to gravity is g. Assume that the rotation of the Earth can be neglected and that the only force acting is Earth's gravity.

(a) Under the approximations that gravity acts everywhere parallel to the line SW and that the acceleration due to Earth's gravity is constant throughout the space through which the wrecking ball is travelling, find the speed v_1 with which the wrecking ball hits the White House, in terms of the constants introduced above.

(b) Taking into account the fact that gravity is non-uniform and acts toward the centre of the Earth, find the speed v_2 with which the wrecking ball hits the White House in terms of the constants introduced above and R, where R is the radius of the Earth, which you may assume is exactly spherical.

(c) Finally, show that

$$v_2 = v_1 \left(1 + (A + B\cos\theta_0)\frac{l}{R} + O\left(\frac{l^2}{R^2}\right)\right),$$

where A and B are constants, which you should determine.

10A Dynamics and Relativity

(a) A particle of positive charge q moves with velocity $\dot{\mathbf{x}}$ in a region in which the magnetic field $\mathbf{B} = (0, 0, B)$ is constant and no other forces act, where $B > 0$. Initially, the particle is at position $\mathbf{x} = (1, 0, 0)$ and $\dot{\mathbf{x}} = (0, v, v)$. Write the equation of motion of the particle and then solve it to find $\mathbf{x}$ as a function of time t. Sketch its path in (x, y, z).

(b) For $B = 0$, three point particles, each of charge q, are fixed at $(0, a/\sqrt{3}, 0)$, $(a/2, -a/(2\sqrt{3}), 0)$ and $(-a/2, -a/(2\sqrt{3}), 0)$, respectively. Another point particle of mass m and charge q is constrained to move in the $z = 0$ plane and suffers Coulomb repulsion from each fixed charge. Neglecting any magnetic fields,

(i) Find the position of an equilibrium point.

(ii) By finding the form of the electric potential near this point, deduce that the equilibrium is stable.

(iii) Consider small displacements of the point particle from the equilibrium point. By resolving forces in the directions $(1, 0, 0)$ and $(0, 1, 0)$, show that the frequency of oscillation is

$$\omega = A\frac{|q|}{\sqrt{m\epsilon_0 a^3}},$$

where A is a constant which you should find.

[You may assume that if two identical charges q are separated by a distance d then the repulsive Coulomb force experienced by each of the charges is $q^2/(4\pi\epsilon_0 d^2)$, where ϵ_0 is a constant.]

11A Dynamics and Relativity

(a) Writing a mass dimension as M, a time dimension as T, a length dimension as L and a charge dimension as Q, write, using relations that you know, the dimensions of:
(i) force
(ii) electric field

(b) In the Large Hadron Collider at CERN, a proton of rest mass m and charge $q > 0$ is accelerated by a constant electric field $\mathbf{E} \neq \mathbf{0}$. At time $t = 0$, the particle is at rest at the origin.

Writing the proton's position as $\mathbf{x}(t)$ and including relativistic effects, calculate $\dot{\mathbf{x}}(t)$. Use your answers to part (a) to check that the dimensions in your expression are correct.

Sketch a graph of $|\dot{\mathbf{x}}(t)|$ versus t, commenting on the $t \to \infty$ limit.

Calculate $|\mathbf{x}(t)|$ as an explicit function of t and find the non-relativistic limit at small times t. What kind of motion is this?

(c) At a later time t_0, an observer in the laboratory frame sees a cosmic microwave photon of energy E_γ hit the accelerated proton, leaving only a Δ^+ particle of mass m_Δ in the final state. In its rest frame, the Δ^+ takes a time t_Δ to decay. How long does it take to decay in the laboratory frame as a function of $q, \mathbf{E}, t_0, m, E_\gamma, m_\Delta, t_\Delta$ and c, the speed of light in a vacuum?

12A Dynamics and Relativity

An inertial frame S and another reference frame S' have a common origin O, and S' rotates with angular velocity vector $\boldsymbol{\omega}(t)$ with respect to S. Derive the results (a) and (b) below, where dot denotes a derivative with respect to time t:

(a) The rates of change of an arbitrary vector $\mathbf{a}(t)$ in frames S and S' are related by

$$(\dot{\mathbf{a}})_S = (\dot{\mathbf{a}})_{S'} + \boldsymbol{\omega} \times \mathbf{a}.$$

(b) The accelerations in S and S' are related by

$$(\ddot{\mathbf{r}})_S = (\ddot{\mathbf{r}})_{S'} + 2\boldsymbol{\omega} \times (\dot{\mathbf{r}})_{S'} + (\dot{\boldsymbol{\omega}})_{S'} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}),$$

where $\mathbf{r}(t)$ is the position vector relative to O.

Just after passing the South Pole, a ski-doo of mass m is travelling on a constant longitude with speed v. Find the magnitude and direction of the sideways component of apparent force experienced by the ski-doo. [The sideways component is locally along the surface of the Earth and perpendicular to the motion of the ski-doo.]

END OF PAPER