UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday, 8 June, 2018 9:00 am to 12:00 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, …, ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Show that if a continued fraction is periodic, then it represents a quadratic irrational. What number is represented by the continued fraction $[7, 7, 7, \ldots]$?

Compute the continued fraction expansion of $\sqrt{23}$. Hence or otherwise find a solution in positive integers to the equation $x^2 - 23y^2 = 1$.

2F Topics in Analysis

Let $0 \leqslant \alpha < 1$ and $A > 0$. If we have an infinite sequence of integers m_n with $1 \leqslant m_n \leqslant An^\alpha$, show that

$$\sum_{n=1}^{\infty} \frac{m_n}{n!}$$

is irrational.

Does the result remain true if the m_n are not restricted to integer values? Justify your answer.

3H Coding & Cryptography

What is a *linear feedback shift register*? Explain the Berlekamp–Massey method for recovering a feedback polynomial of a linear feedback shift register from its output. Illustrate the method in the case when we observe output

$$0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0 \ldots.$$

4G Automata and Formal Languages

(a) State the *s-m-n* theorem, the recursion theorem, and Rice's theorem.

(b) Show that if $g : \mathbb{N}^2 \to \mathbb{N}$ is partial recursive, then there is some $e \in \mathbb{N}$ such that

$$f_{e,1}(y) = g(e, y) \quad \forall y \in \mathbb{N}.$$

(c) By considering the partial function $g : \mathbb{N}^2 \to \mathbb{N}$ given by

$$g(x,y) = \begin{cases} 0 & \text{if } y < x \\ \uparrow & \text{otherwise,} \end{cases}$$

show there exists some $m \in \mathbb{N}$ such that W_m has exactly m elements.

(d) Given $n \in \mathbb{N}$, is it possible to compute whether or not W_n has exactly 9 elements? Justify your answer.

[*Note that we define* $\mathbb{N} = \{0, 1, \ldots\}$. *Any use of Church's thesis in your answers should be explicitly stated.*]

5J Statistical Modelling

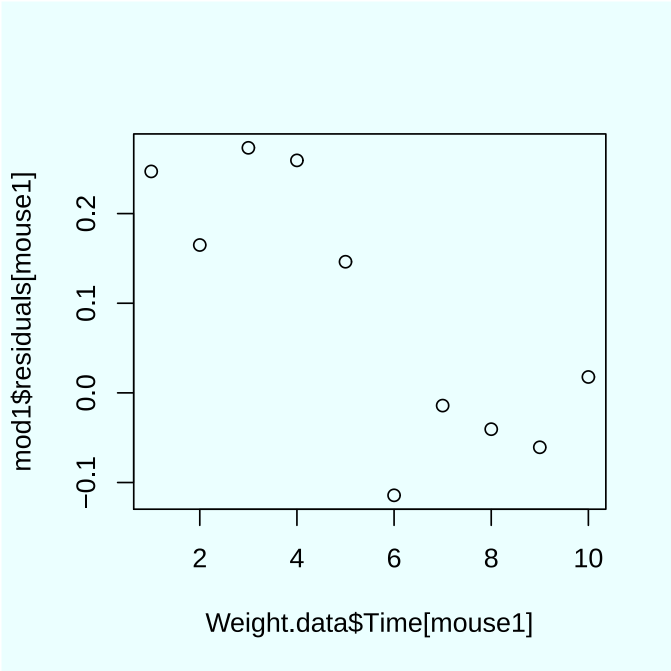
A scientist is studying the effects of a drug on the weight of mice. Forty mice are divided into two groups, control and treatment. The mice in the treatment group are given the drug, and those in the control group are given water instead. The mice are kept in 8 different cages. The weight of each mouse is monitored for 10 days, and the results of the experiment are recorded in the data frame `Weight.data`. Consider the following R code and its output.

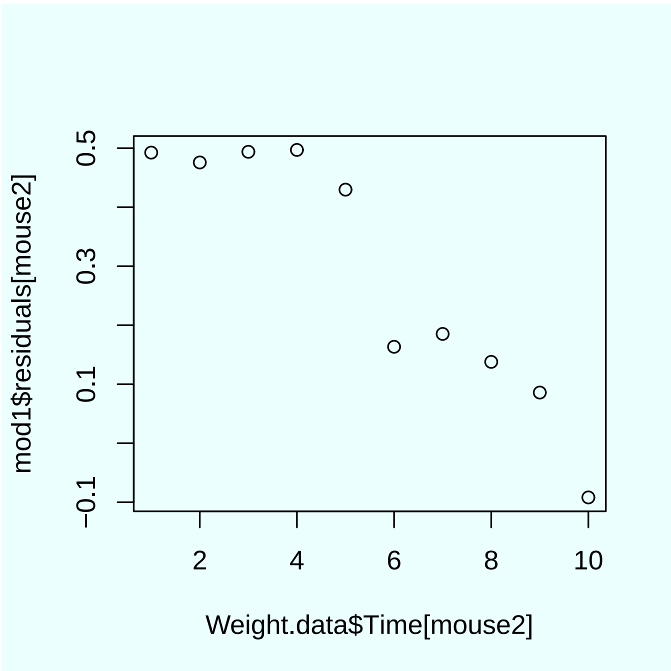
```
> head(Weight.data)
  Time   Group Cage Mouse   Weight
1    1 Control    1     1 24.77578
2    2 Control    1     1 24.68766
3    3 Control    1     1 24.79008
4    4 Control    1     1 24.77005
5    5 Control    1     1 24.65092
6    6 Control    1     1 24.38436
> mod1 = lm(Weight ~ Time*Group + Cage, data=Weight.data)
> summary(mod1)

Call:
lm(formula = Weight ~ Time * Group + Cage, data = Weight.data)

Residuals:
     Min       1Q   Median       3Q      Max
-1.36903 -0.33527 -0.01719  0.38807  1.24368

Coefficients:
                     Estimate Std. Error t value Pr(>|t|)
(Intercept)         24.534771   0.100336 244.525  < 2e-16 ***
Time                -0.006023   0.012616  -0.477  0.63334
GroupTreatment       0.321837   0.121993   2.638  0.00867 **
Cage2               -0.400228   0.095875  -4.174 3.68e-05 ***
Cage3                0.286941   0.102494   2.800  0.00537 **
Cage4                0.007535   0.095875   0.079  0.93740
Cage6                0.124767   0.125530   0.994  0.32087
Cage8               -0.295168   0.125530  -2.351  0.01920 *
Time:GroupTreatment -0.173515   0.017842  -9.725  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.5125 on 391 degrees of freedom
Multiple R-squared:  0.5591,Adjusted R-squared:   0.55
F-statistic: 61.97 on 8 and 391 DF,  p-value: < 2.2e-16
```

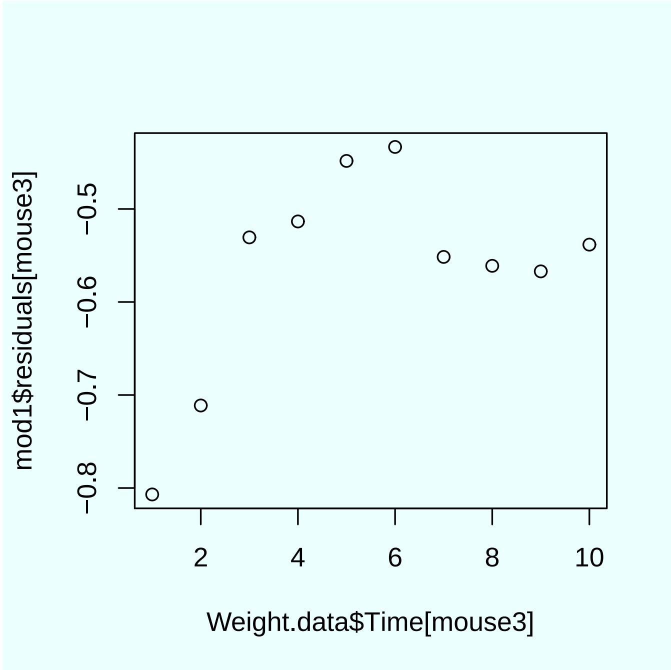
Which parameters describe the rate of weight loss with time in each group? According to the R output, is there a statistically significant weight loss with time in the control group?

Three diagnostic plots were generated using the following R code.

```
mouse1 = (Weight.data$Mouse==1)
plot(Weight.data$Time[mouse1],mod1$residuals[mouse1])
mouse2 = (Weight.data$Mouse==2)
plot(Weight.data$Time[mouse2],mod1$residuals[mouse2])
mouse3 = (Weight.data$Mouse==3)
plot(Weight.data$Time[mouse3],mod1$residuals[mouse3])
```







Based on these plots, should you trust the significance tests shown in the output of the command `summary(mod1)`? Explain.

6C Mathematical Biology

Consider a model of a population N_τ in discrete time

$$N_{\tau+1} = \frac{rN_\tau}{(1+bN_\tau)^2},$$

where $r, b > 0$ are constants and $\tau = 1, 2, 3, \ldots$. Interpret the constants and show that for $r > 1$ there is a stable fixed point.

Suppose the initial condition is $N_1 = 1/b$ and that $r > 4$. Show, using a cobweb diagram, that the population N_τ is bounded as

$$\frac{4r^2}{(4+r)^2 b} \leqslant N_\tau \leqslant \frac{r}{4b}$$

and attains the bounds.

7B Further Complex Methods

State the conditions for a point $z = z_0$ to be a *regular singular point* of a linear second-order homogeneous ordinary differential equation in the complex plane.

Find all singular points of the Bessel equation

$$z^2 y''(z) + z y'(z) + \left(z^2 - \frac{1}{4}\right) y(z) = 0\,, \qquad (*)$$

and determine whether they are regular or irregular.

By writing $y(z) = f(z)/\sqrt{z}$, find two linearly independent solutions of $(*)$. Comment on the relationship of your solutions to the nature of the singular points.

8B Classical Dynamics

State and prove Noether's theorem in Lagrangian mechanics.

Consider a Lagrangian

$$\mathcal{L} = \frac{1}{2}\frac{\dot{x}^2 + \dot{y}^2}{y^2} - V\left(\frac{x}{y}\right)$$

for a particle moving in the upper half-plane $\{(x, y) \in \mathbb{R}^2, y > 0\}$ in a potential V which only depends on x/y. Find two independent first integrals.

9B Cosmology

A constant overdensity is created by taking a spherical region of a flat matter-dominated universe with radius $\overline{R}$ and compressing it into a region with radius $R < \overline{R}$. The evolution is governed by the parametric equations

$$R = AR_0(1 - \cos\theta)\,, \quad t = B(\theta - \sin\theta),$$

where R_0 is a constant and

$$A = \frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)}\,, \quad B = \frac{\Omega_{m,0}}{2H_0\,(\Omega_{m,0} - 1)^{3/2}}\,,$$

where H_0 is the Hubble constant and $\Omega_{m,0}$ is the fractional overdensity at time t_0.

Show that, as $t \to 0^+$,

$$R(t) = R_0\,\Omega_{m,0}^{1/3}\,a(t)\left(1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3} + \ldots\right)\,,$$

where the scale factor is given by $a(t) = (3H_0t/2)^{2/3}$.

Show that, at the linear level, the density perturbation δ_{linear} grows as $a(t)$. Show that, when the spherical overdensity has collapsed to zero radius, the linear perturbation has value $\delta_{\text{linear}} = \frac{3}{20}\,(12\pi)^{2/3}$.

10D Quantum Information and Computation

Let B_n denote the set of all n-bit strings. Suppose we are given a 2-qubit quantum gate I_{x_0} which is promised to be of the form

$$I_{x_0} \,|\, x \rangle = \begin{cases} \;\;\,|\, x \rangle & x \neq x_0 \\ -\,|\, x \rangle & x = x_0 \end{cases}$$

but the 2-bit string x_0 is unknown to us. We wish to determine x_0 with the least number of queries to I_{x_0}. Define $A = I - 2\,|\,\psi\rangle\langle\psi|$, where I is the identity operator and $|\,\psi\rangle = \frac{1}{2}\sum_{x\in B_2} |\,x\rangle$.

(a) Is A unitary? Justify your answer.

(b) Compute the action of I_{x_0} on $|\,\psi\rangle$, and the action of $|\,\psi\rangle\langle\psi|$ on $|\,x_0\rangle$, in each case expressing your answer in terms of $|\,\psi\rangle$ and $|\,x_0\rangle$. Hence or otherwise show that x_0 may be determined with certainty using only one application of the gate I_{x_0}, together with any other gates that are independent of x_0.

(c) Let $f_{x_0} : B_2 \to B_1$ be the function having value 0 for all $x \neq x_0$ and having value 1 for $x = x_0$. It is known that a single use of I_{x_0} can be implemented with a single query to a quantum oracle for the function f_{x_0}. But suppose instead that we have a classical oracle for f_{x_0}, *i.e.* a black box which, on input x, outputs the value of $f_{x_0}(x)$. Can we determine x_0 with certainty using a single query to the classical oracle? Justify your answer.

SECTION II

11G Number Theory

(a) State and prove the Fermat–Euler theorem. Let p be a prime and k a positive integer. Show that $b^k \equiv b \pmod{p}$ holds for every integer b if and only if $k \equiv 1 \pmod{p-1}$.

(b) Let $N \geqslant 3$ be an odd integer and b be an integer with $(b, N) = 1$. What does it mean to say that N is a *Fermat pseudoprime to base* b? What does it mean to say that N is a *Carmichael number*?

Show that every Carmichael number is squarefree, and that if N is squarefree, then N is a Carmichael number if and only if $N \equiv 1 \pmod{p-1}$ for every prime divisor p of N. Deduce that a Carmichael number is a product of at least three primes.

(c) Let r be a fixed odd prime. Show that there are only finitely many pairs of primes p, q for which $N = pqr$ is a Carmichael number.

[*You may assume throughout that* $(\mathbb{Z}/p^n\mathbb{Z})^*$ *is cyclic for every odd prime* p *and every integer* $n \geqslant 1$.]

12F Topics in Analysis

We work in $\mathbb{C}$. Consider

$$K = \{z : |z-2| \leqslant 1\} \cup \{z : |z+2| \leqslant 1\}$$

and

$$\Omega = \{z : |z-2| < 3/2\} \cup \{z : |z+2| < 3/2\}.$$

Show that if $f : \Omega \to \mathbb{C}$ is analytic, then there is a sequence of polynomials p_n such that $p_n(z) \to f(z)$ uniformly on K.

Show that there is a sequence of polynomials P_n such that $P_n(z) \to 0$ uniformly for $|z-2| \leqslant 1$ and $P_n(z) \to 1$ uniformly for $|z+2| \leqslant 1$.

Give two disjoint non-empty bounded closed sets K_1 and K_2 such that there does not exist a sequence of polynomials Q_n with $Q_n(z) \to 0$ uniformly on K_1 and $Q_n(z) \to 1$ uniformly on K_2. Justify your answer.

13J Statistical Modelling

Bridge is a card game played by 2 teams of 2 players each. A bridge club records the outcomes of many games between teams formed by its m members. The outcomes are modelled by

$$\mathbb{P}(\text{team } \{i,j\} \text{ wins against team } \{k,\ell\}) = \frac{\exp(\beta_i + \beta_j + \beta_{\{i,j\}} - \beta_k - \beta_\ell - \beta_{\{k,\ell\}})}{1 + \exp(\beta_i + \beta_j + \beta_{\{i,j\}} - \beta_k - \beta_\ell - \beta_{\{k,\ell\}})},$$

where $\beta_i \in \mathbb{R}$ is a parameter representing the skill of player i, and $\beta_{\{i,j\}} \in \mathbb{R}$ is a parameter representing how well-matched the team formed by i and j is.

(a) Would it make sense to include an intercept in this logistic regression model? Explain your answer.

(b) Suppose that players 1 and 2 always play together as a team. Is there a unique maximum likelihood estimate for the parameters β_1, β_2 and $\beta_{\{1,2\}}$? Explain your answer.

(c) Under the model defined above, derive the asymptotic distribution (including the values of all relevant parameters) for the maximum likelihood estimate of the probability that team $\{i,j\}$ wins a game against team $\{k,\ell\}$. You can state it as a function of the true vector of parameters β, and the Fisher information matrix $i_N(\beta)$ with N games. You may assume that $i_N(\beta)/N \to I(\beta)$ as $N \to \infty$, and that β has a unique maximum likelihood estimate for N large enough.

14C Mathematical Biology

An activator-inhibitor reaction diffusion system is given, in dimensionless form, by

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + \frac{u^2}{v} - 2bu, \qquad \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + u^2 - v,$$

where d and b are positive constants. Which symbol represents the *concentration of activator* and which the *inhibitor*? Determine the positive steady states and show, by an examination of the eigenvalues in a linear stability analysis of the spatially uniform situation, that the reaction kinetics are stable if $b < \frac{1}{2}$.

Determine the conditions for the steady state to be driven unstable by diffusion, and sketch the (b,d) parameter space in which the diffusion-driven instability occurs. Find the critical wavenumber k_c at the bifurcation to such a diffusion-driven instability.

15B Classical Dynamics

Given a Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t)$ with degrees of freedom q_i, define the *Hamiltonian* and show how Hamilton's equations arise from the Lagrange equations and the Legendre transform.

Consider the Lagrangian for a symmetric top moving in constant gravity:

$$\mathcal{L} = \frac{1}{2}A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}B(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta,$$

where A, B, M, g and l are constants. Construct the corresponding Hamiltonian, and find three independent Poisson-commuting first integrals of Hamilton's equations.

16G Logic and Set Theory

State and prove the ϵ-Recursion Theorem. [You may assume the Principle of ϵ-Induction.]

What does it mean to say that a relation r on a set x is *well-founded* and *extensional*? State and prove Mostowski's Collapsing Theorem. [You may use any recursion theorem from the course, provided you state it precisely.]

For which sets x is it the case that every well-founded extensional relation on x is isomorphic to the relation ϵ on some transitive subset of V_ω?

17I Graph Theory

Let $s \geqslant 3$. Define the *Ramsey number* $R(s)$. Show that $R(s)$ exists and that $R(s) \leqslant 4^s$.

Show that $R(3) = 6$. Show that (up to relabelling the vertices) there is a unique way to colour the edges of the complete graph K_5 blue and yellow with no monochromatic triangle.

What is the least positive integer n such that the edges of the complete graph K_6 can be coloured blue and yellow in such a way that there are precisely n monochromatic triangles?

18I Galois Theory

Let K be a field of characteristic $p > 0$ and let L be the splitting field of the polynomial $f(t) = t^p - t + a$ over K, where $a \in K$. Let $\alpha \in L$ be a root of $f(t)$.

If $L \neq K$, show that $f(t)$ is irreducible over K, that $L = K(\alpha)$, and that L is a Galois extension of K. What is $\mathrm{Gal}(L/K)$?

19I Representation Theory

Define G =SU(2) and write down a complete list

$$\{V_n : n = 0, 1, 2, \dots\}$$

of its continuous finite-dimensional irreducible representations. You should define all the terms you use but proofs are not required. Find the character χ_{V_n} of V_n. State the Clebsch–Gordan formula.

(a) Stating clearly any properties of symmetric powers that you need, decompose the following spaces into irreducible representations of G:

(i) $V_4 \otimes V_3$, $V_3 \otimes V_3$, $S^2 V_3$;

(ii) $V_1 \otimes \cdots \otimes V_1$ (with n multiplicands);

(iii) $S^3 V_2$.

(b) Let G act on the space $M_3(\mathbb{C})$ of 3×3 complex matrices by

$$A : X \mapsto A_1 X A_1^{-1},$$

where A_1 is the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Show that this gives a representation of G and decompose it into irreducible summands.

20G Number Fields

Let $m \geqslant 2$ be a square-free integer, and let $n \geqslant 2$ be an integer. Let $L = \mathbb{Q}(\sqrt[n]{m})$.

(a) By considering the factorisation of (m) into prime ideals, show that $[L : \mathbb{Q}] = n$.

(b) Let $T : L \times L \to \mathbb{Q}$ be the bilinear form defined by $T(x, y) = \operatorname{tr}_{L/\mathbb{Q}}(xy)$. Let $\beta_i = \sqrt[n]{m}^{\,i}$, $i = 0, \dots, n-1$. Calculate the dual basis $\beta_0^*, \dots, \beta_{n-1}^*$ of L with respect to T, and deduce that $\mathcal{O}_L \subset \frac{1}{nm}\mathbb{Z}[\sqrt[n]{m}]$.

(c) Show that if p is a prime and $n = m = p$, then $\mathcal{O}_L = \mathbb{Z}[\sqrt[p]{p}]$.

21H Algebraic Topology

(a) State the Mayer–Vietoris theorem for a union of simplicial complexes

$$K = M \cup N$$

with $L = M \cap N$.

(b) Construct the map $\partial_* : H_k(K) \to H_{k-1}(L)$ that appears in the statement of the theorem. [You do not need to prove that the map is well defined, or a homomorphism.]

(c) Let K be a simplicial complex with $|K|$ homeomorphic to the n-dimensional sphere S^n, for $n \geqslant 2$. Let $M \subseteq K$ be a subcomplex with $|M|$ homeomorphic to $S^{n-1} \times [-1,1]$. Suppose that $K = M \cup N$, such that $L = M \cap N$ has polyhedron $|L|$ identified with $S^{n-1} \times \{-1,1\} \subseteq S^{n-1} \times [-1,1]$. Prove that $|N|$ has two path components.

22F Linear Analysis

(a) Let X be a separable normed space. For any sequence $(f_n)_{n\in\mathbb{N}} \subset X^*$ with $\|f_n\| \leqslant 1$ for all n, show that there is $f \in X^*$ and a subsequence $\Lambda \subset \mathbb{N}$ such that $f_n(x) \to f(x)$ for all $x \in X$ as $n \in \Lambda$, $n \to \infty$. [You may use without proof the fact that X^* is complete and that any bounded linear map $f : D \to \mathbb{R}$, where $D \subset X$ is a dense linear subspace, can be extended uniquely to an element $f \in X^*$.]

(b) Let H be a Hilbert space and $U : H \to H$ a unitary map. Let

$$I = \{x \in H : Ux = x\}, \qquad W = \{Ux - x : x \in H\}.$$

Prove that I and W are orthogonal, $H = I \oplus \overline{W}$, and that for every $x \in H$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i x = Px,$$

where P is the orthogonal projection onto the closed subspace I.

(c) Let $T : C(S^1) \to C(S^1)$ be a linear map, where $S^1 = \{e^{i\theta} \in \mathbb{C} : \theta \in \mathbb{R}\}$ is the unit circle, induced by a homeomorphism $\tau : S^1 \to S^1$ by $(Tf)e^{i\theta} = f(\tau(e^{i\theta}))$. Prove that there exists $\mu \in C(S^1)^*$ with $\mu(1_{S^1}) = 1$ such that $\mu(Tf) = \mu(f)$ for all $f \in C(S^1)$. (Here 1_{S^1} denotes the function on S^1 which returns 1 identically.) If T is not the identity map, does it follow that μ as above is necessarily unique? Justify your answer.

23F Analysis of Functions

Here and below, $\Phi : \mathbb{R} \to \mathbb{R}$ is smooth such that $\int_{\mathbb{R}} e^{-\Phi(x)}\,\mathrm{d}x = 1$ and

$$\lim_{|x|\to+\infty}\left(\frac{|\Phi'(x)|^2}{4} - \frac{\Phi''(x)}{2}\right) = \ell \in (0,+\infty).$$

$C_c^1(\mathbb{R})$ denotes the set of continuously differentiable complex-valued functions with compact support on $\mathbb{R}$.

(a) Prove that there are constants $R_0 > 0$, $\lambda_1 > 0$ and $K_1 > 0$ so that for any $R \geqslant R_0$ and $h \in C_c^1(\mathbb{R})$:

$$\int_{\mathbb{R}} \left|h'(x)\right|^2 e^{-\Phi(x)}\,dx \geqslant \lambda_1 \int_{\{|x|\geqslant R\}} |h(x)|^2 e^{-\Phi(x)}\,dx - K_1 \int_{\{|x|\leqslant R\}} |h(x)|^2 e^{-\Phi(x)}\,dx.$$

[*Hint: Denote $g := he^{-\Phi/2}$, expand the square and integrate by parts.*]

(b) Prove that, given any $R > 0$, there is a $C_R > 0$ so that for any $h \in C^1([-R,R])$ with $\int_{-R}^{+R} h(x)e^{-\Phi(x)}dx = 0$:

$$\max_{x\in[-R,R]} |h(x)| + \sup_{\{x,y\in[-R,R],\ x\neq y\}} \frac{|h(x)-h(y)|}{|x-y|^{1/2}} \leqslant C_R \left(\int_{-R}^{+R} |h'(x)|^2 e^{-\Phi(x)}\,dx\right)^{1/2}.$$

[*Hint: Use the fundamental theorem of calculus to control the second term of the left-hand side, and then compare h to its weighted mean to control the first term of the left-hand side.*]

(c) Prove that, given any $R > 0$, there is a $\lambda_R > 0$ so that for any $h \in C^1([-R,R])$:

$$\int_{-R}^{+R} \left|h'(x)\right|^2 e^{-\Phi(x)}\,dx \geqslant \lambda_R \int_{-R}^{+R} \left| h(x) - \frac{\int_{-R}^{+R} h(y)e^{-\Phi(y)}\,dy}{\int_{-R}^{+R} e^{-\Phi(y)}\,dy}\right|^2 e^{-\Phi(x)}\,dx.$$

[*Hint: Show first that one can reduce to the case $\int_{-R}^{+R} he^{-\Phi} = 0$. Then argue by contradiction with the help of the Arzelà–Ascoli theorem and part (b).*]

(d) Deduce that there is a $\lambda_0 > 0$ so that for any $h \in C_c^1(\mathbb{R})$:

$$\int_{\mathbb{R}} \left|h'(x)\right|^2 e^{-\Phi(x)}\,dx \geqslant \lambda_0 \int_{\mathbb{R}} \left| h(x) - \left(\int_{\mathbb{R}} h(y)e^{-\Phi(y)}\,dy\right)\right|^2 e^{-\Phi(x)}\,dx.$$

[*Hint: Show first that one can reduce to the case $\int_{\mathbb{R}} he^{-\Phi} = 0$. Then combine the inequality (a), multiplied by a constant of the form $\epsilon = \epsilon_0\lambda_R$ (where $\epsilon_0 > 0$ is chosen so that ϵ be sufficiently small), and the inequality (c).*]

24I Algebraic Geometry

State a theorem which describes the canonical divisor of a smooth plane curve C in terms of the divisor of a hyperplane section. Express the degree of the canonical divisor K_C and the genus of C in terms of the degree of C. [You need not prove these statements.]

From now on, we work over $\mathbb{C}$. Consider the curve in $\mathbf{A}^2$ defined by the equation

$$y + x^3 + xy^3 = 0.$$

Let C be its projective completion. Show that C is smooth.

Compute the genus of C by applying the Riemann–Hurwitz theorem to the morphism $C \to \mathbf{P}^1$ induced from the rational map $(x, y) \mapsto y$. [You may assume that the discriminant of $x^3 + ax + b$ is $-4a^3 - 27b^2$.]

25I Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface.

(a) Define what it means for a curve $\gamma : I \to S$ to be a *geodesic*, where $I = (a, b)$ and $-\infty \leqslant a < b \leqslant \infty$.

(b) A geodesic $\gamma : I \to S$ is said to be *maximal* if any geodesic $\widetilde{\gamma} : \widetilde{I} \to S$ with $I \subset \widetilde{I}$ and $\widetilde{\gamma}|_I = \gamma$ satisfies $I = \widetilde{I}$. A surface is said to be *geodesically complete* if all maximal geodesics are defined on $I = (-\infty, \infty)$, otherwise, the surface is said to be *geodesically incomplete*. Give an example, with justification, of a non-compact geodesically complete surface S which is not a plane.

(c) Assume that along any maximal geodesic

$$\gamma : (-T_-, T_+) \to S,$$

the following holds:

$$T_\pm < \infty \implies \limsup_{s \to T_\pm} |K(\gamma(\pm s))| = \infty. \qquad (*)$$

Here K denotes the Gaussian curvature of S.

(i) Show that S is *inextendible*, i.e. if $\widetilde{S} \subset \mathbb{R}^3$ is a connected surface with $S \subset \widetilde{S}$, then $\widetilde{S} = S$.

(ii) Give an example of a surface S which is geodesically incomplete and satisfies $(*)$. Do all geodesically incomplete inextendible surfaces satisfy $(*)$? Justify your answer.

[*You may use facts about geodesics from the course provided they are clearly stated.*]

26J Probability and Measure

Let $(X, \mathcal{A})$ be a measurable space. Let $T : X \to X$ be a measurable map, and μ a probability measure on $(X, \mathcal{A})$.

(a) State the definition of the following properties of the system $(X, \mathcal{A}, \mu, T)$:

(i) μ is *T-invariant*.

(ii) T is *ergodic* with respect to μ.

(b) State the pointwise ergodic theorem.

(c) Give an example of a probability measure preserving system $(X, \mathcal{A}, \mu, T)$ in which $\mathrm{Card}(T^{-1}\{x\}) > 1$ for μ-a.e. x.

(d) Assume X is finite and $\mathcal{A}$ is the boolean algebra of all subsets of X. Suppose that μ is a T-invariant probability measure on X such that $\mu(\{x\}) > 0$ for all $x \in X$. Show that T is a bijection.

(e) Let $X = \mathbb{N}$, the set of positive integers, and $\mathcal{A}$ be the σ-algebra of all subsets of X. Suppose that μ is a T-invariant ergodic probability measure on X. Show that there is a finite subset $Y \subseteq X$ with $\mu(Y) = 1$.

27J Applied Probability

Let $X_1, X_2, \dots$ be independent, identically distributed random variables with finite mean μ. Explain what is meant by saying that the random variable M is a *stopping time* with respect to the sequence $(X_i : i = 1, 2, \dots)$.

Let M be a stopping time with finite mean $\mathbb{E}(M)$. Prove Wald's equation:

$$\mathbb{E}\left(\sum_{i=1}^{M} X_i\right) = \mu \mathbb{E}(M).$$

[Here and in the following, you may use any standard theorem about integration.]

Suppose the X_i are strictly positive, and let N be the renewal process with interarrival times $(X_i : i = 1, 2, \dots)$. Prove that $m(t) = \mathbb{E}(N_t)$ satisfies the elementary renewal theorem:

$$\frac{1}{t} m(t) \to \frac{1}{\mu} \qquad \text{as } t \to \infty.$$

A computer keyboard contains 100 different keys, including the lower and upper case letters, the usual symbols, and the space bar. A monkey taps the keys uniformly at random. Find the mean number of keys tapped until the first appearance of the sequence 'lava' as a sequence of 4 consecutive characters.

Find the mean number of keys tapped until the first appearance of the sequence 'aa' as a sequence of 2 consecutive characters.

28K Principles of Statistics

Let $g : \mathbb{R} \to \mathbb{R}$ be an unknown function, twice continuously differentiable with $|g''(x)| \leqslant M$ for all $x \in \mathbb{R}$. For some $x_0 \in \mathbb{R}$, we know the value $g(x_0)$ and we wish to estimate its derivative $g'(x_0)$. To do so, we have access to a pseudo-random number generator that gives $U_1^*, \ldots, U_N^*$ i.i.d. uniform over $[0,1]$, and a machine that takes input $x_1, \ldots, x_N \in \mathbb{R}$ and returns $g(x_i) + \varepsilon_i$, where the ε_i are i.i.d. $\mathcal{N}(0, \sigma^2)$.

(a) Explain how this setup allows us to generate N independent $X_i = x_0 + hZ_i$, where the Z_i take value 1 or -1 with probability $1/2$, for any $h > 0$.

(b) We denote by Y_i the output $g(X_i) + \varepsilon_i$. Show that for some independent $\xi_i \in \mathbb{R}$

$$Y_i - g(x_0) = hZ_i\, g'(x_0) + \frac{h^2}{2} g''(\xi_i) + \varepsilon_i \,.$$

(c) Using the intuition given by the least-squares estimator, justify the use of the estimator $\hat{g}_N$ given by

$$\hat{g}_N = \frac{1}{N} \sum_{i=1}^{N} \frac{Z_i(Y_i - g(x_0))}{h} \,.$$

(d) Show that

$$\mathbb{E}[|\hat{g}_N - g'(x_0)|^2] \leqslant \frac{h^2 M^2}{4} + \frac{\sigma^2}{N h^2} \,.$$

Show that for some choice h_N of parameter h, this implies

$$\mathbb{E}[|\hat{g}_N - g'(x_0)|^2] \leqslant \frac{\sigma M}{\sqrt{N}} \,.$$

29K Stochastic Financial Models

Consider a utility function $U : \mathbb{R} \to \mathbb{R}$, which is assumed to be concave, strictly increasing and twice differentiable. Further, U satisfies

$$\bigl|U'(x)\bigr| \leqslant c\,|x|^{\alpha}, \qquad \forall x \in \mathbb{R},$$

for some positive constants c and α. Let X be an $\mathcal{N}(\mu, \sigma^2)$-distributed random variable and set $f(\mu, \sigma) := \mathbb{E}[U(X)]$.

(a) Show that

$$\mathbb{E}\bigl[U'(X)(X-\mu)\bigr] = \sigma^2\,\mathbb{E}\bigl[U''(X)\bigr].$$

(b) Show that $\frac{\partial f}{\partial \mu} > 0$ and $\frac{\partial f}{\partial \sigma} \leqslant 0$. Discuss this result in the context of mean-variance analysis.

(c) Show that f is concave in μ and σ, i.e. check that the matrix of second derivatives is negative semi-definite. [You may use without proof the fact that if a $2{\times}2$ matrix has non-positive diagonal entries and a non-negative determinant, then it is negative semi-definite.]

30K Optimisation and Control

Consider the deterministic system

$$\dot{x}_t = u_t$$

where x_t and u_t are scalars. Here x_t is the state variable and the control variable u_t is to be chosen to minimise, for a fixed $h > 0$, the cost

$$x_h^2 + \int_0^h c_t u_t^2 \mathrm{d}t,$$

where c_t is known and $c_t > c > 0$ for all t. Let $F(x,t)$ be the minimal cost from state x and time t.

(a) By writing the dynamic programming equation in infinitesimal form and taking the appropriate limit show that $F(x,t)$ satisfies

$$\frac{\partial F}{\partial t} = -\inf_u \left[c_t u^2 + \frac{\partial F}{\partial x} u\right], \quad t < h$$

with boundary condition $F(x,h) = x^2$.

(b) Determine the form of the optimal control in the special case where c_t is constant, and also in general.

31B Asymptotic Methods

Show that

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\,d\theta$$

is a solution to the equation

$$xy'' + y' - xy = 0\,,$$

and obtain the first two terms in the asymptotic expansion of $I_0(x)$ as $x \to +\infty$.

For $x > 0$, define a new dependent variable $w(x) = x^{\frac{1}{2}}y(x)$, and show that if y solves the preceding equation then

$$w'' + \left(\frac{1}{4x^2} - 1\right)w = 0\,.$$

Obtain the Liouville–Green approximate solutions to this equation for large positive x, and compare with your asymptotic expansion for $I_0(x)$ at the leading order.

32E Dynamical Systems

Let $F : I \to I$ be a continuous one-dimensional map of an interval $I \subset \mathbb{R}$. Define what it means (i) for F to have a *horseshoe* (ii) for F to be *chaotic*. [Glendinning's definition should be used throughout this question.]

Prove that if F has a 3-cycle $x_1 < x_2 < x_3$ then F is chaotic. [You may assume the intermediate value theorem and any corollaries of it.]

State Sharkovsky's theorem.

Use the above results to deduce that if F has an N-cycle, where N is any integer that is not a power of 2, then F is chaotic.

Explain briefly why if F is chaotic then F has N-cycles for many values of N that are not powers of 2. [You may assume that a map with a horseshoe acts on some set Λ like the Bernoulli shift map acts on [0,1).]

The logistic map is not chaotic when $\mu < \mu_\infty \approx 3.57$ and it has 3-cycles when $\mu > 1 + \sqrt{8} \approx 3.84$. What can be deduced from these statements about the values of μ for which the logistic map has a 10-cycle?

33D Principles of Quantum Mechanics

The spin operators obey the commutation relations $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$. Let $|s,\sigma\rangle$ be an eigenstate of the spin operators S_z and $\mathbf{S}^2$, with $S_z|s,\sigma\rangle = \sigma\hbar\,|s,\sigma\rangle$ and $\mathbf{S}^2|s,\sigma\rangle = s(s+1)\hbar^2\,|s,\sigma\rangle$. Show that

$$S_{\pm}|s,\sigma\rangle = \sqrt{s(s+1)-\sigma(\sigma\pm 1)}\,\hbar\,|s,\sigma\pm 1\rangle\,,$$

where $S_{\pm} = S_x \pm iS_y$. When $s=1$, use this to derive the explicit matrix representation

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

in a basis in which S_z is diagonal.

A beam of atoms, each with spin 1, is polarised to have spin $+\hbar$ along the direction $\mathbf{n} = (\sin\theta, 0, \cos\theta)$. This beam enters a Stern–Gerlach filter that splits the atoms according to their spin along the $\hat{\mathbf{z}}$-axis. Show that $N_+/N_- = \cot^4(\theta/2)$, where N_+ (respectively, N_-) is the number of atoms emerging from the filter with spins parallel (respectively, anti-parallel) to $\hat{\mathbf{z}}$.

34A Applications of Quantum Mechanics

Define a *Bravais lattice* Λ in three dimensions. Define the *reciprocal lattice* $\Lambda^\star$. Define the *Brillouin zone.*

An FCC lattice has a basis of primitive vectors given by

$$\mathbf{a}_1 = \frac{a}{2}(\mathbf{e}_2 + \mathbf{e}_3)\ ,\quad \mathbf{a}_2 = \frac{a}{2}(\mathbf{e}_1 + \mathbf{e}_3)\ ,\quad \mathbf{a}_3 = \frac{a}{2}(\mathbf{e}_1 + \mathbf{e}_2),$$

where $\mathbf{e}_i$ is an orthonormal basis of $\mathbb{R}^3$. Find a basis of reciprocal lattice vectors. What is the volume of the Brillouin zone?

The asymptotic wavefunction for a particle, of wavevector $\mathbf{k}$, scattering off a potential $V(\mathbf{r})$ is

$$\psi(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}} + f_{\mathrm{V}}(\mathbf{k};\mathbf{k}')\frac{e^{ikr}}{r}\ ,$$

where $\mathbf{k}' = k\hat{\mathbf{r}}$ and $f_{\mathrm{V}}(\mathbf{k};\mathbf{k}')$ is the scattering amplitude. Give a formula for the *Born approximation* to the scattering amplitude.

Scattering of a particle off a single atom is modelled by a potential $V(\mathbf{r}) = V_0\delta(r-d)$ with δ-function support on a spherical shell, $r = |\mathbf{r}| = d$ centred at the origin. Calculate the Born approximation to the scattering amplitude, denoting the resulting expression as $\tilde{f}_{\mathrm{V}}(\mathbf{k};\mathbf{k}')$.

Scattering of a particle off a crystal consisting of atoms located at the vertices of a lattice Λ is modelled by a potential

$$V_\Lambda = \sum_{\mathbf{R}\in\Lambda} V(\mathbf{r}-\mathbf{R}),$$

where $V(\mathbf{r}) = V_0\delta(r-d)$ as above. Calculate the Born approximation to the scattering amplitude giving your answer in terms of your approximate expression $\tilde{f}_{\mathrm{V}}$ for scattering off a single atom. Show that the resulting amplitude vanishes unless the momentum transfer $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ lies in the reciprocal lattice $\Lambda^\star$.

For the particular FCC lattice considered above, show that, when $k = |\mathbf{k}| > 2\pi/a$, scattering occurs for two values of the scattering angle, θ_1 and θ_2, related by

$$\frac{\sin\left(\frac{\theta_1}{2}\right)}{\sin\left(\frac{\theta_2}{2}\right)} = \frac{2}{\sqrt{3}}\,.$$

35A Statistical Physics

The one-dimensional Ising model consists of a set of N spins s_i with Hamiltonian

$$H = -J \sum_{i=1}^{N} s_i s_{i+1} - \frac{B}{2} \sum_{i=1}^{N} (s_i + s_{i+1}),$$

where periodic boundary conditions are imposed so $s_{N+1} = s_1$. Here J is a positive coupling constant and B is an external magnetic field. Define a 2×2 matrix M with elements

$$M_{st} = \exp\left[\beta J st + \frac{\beta B}{2}(s+t)\right],$$

where indices s, t take values ± 1 and $\beta = (kT)^{-1}$ with k Boltzmann's constant and T temperature.

(a) Prove that the partition function of the Ising model can be written as

$$Z = \mathrm{Tr}(M^N).$$

Calculate the eigenvalues of M and hence determine the free energy in the thermodynamic limit $N \to \infty$. Explain why the Ising model does not exhibit a phase transition in one dimension.

(b) Consider the case of zero magnetic field $B = 0$. The correlation function $\langle s_i s_j \rangle$ is defined by

$$\langle s_i s_j \rangle = \frac{1}{Z} \sum_{\{s_k\}} s_i s_j e^{-\beta H}.$$

(i) Show that, for $i > 1$,

$$\langle s_1 s_i \rangle = \frac{1}{Z} \sum_{s,t} st (M^{i-1})_{st} (M^{N-i+1})_{ts}.$$

(ii) By diagonalizing M, or otherwise, calculate M^p for any positive integer p. Hence show that

$$\langle s_1 s_i \rangle = \frac{\tanh^{i-1}(\beta J) + \tanh^{N-i+1}(\beta J)}{1 + \tanh^N(\beta J)}.$$

36D Electrodynamics

(a) Define the *polarisation* of a dielectric material and explain what is meant by the term *bound charge*.

Consider a sample of material with spatially dependent polarisation $\mathbf{P}(\mathbf{x})$ occupying a region V with surface S. Show that, in the absence of free charge, the resulting scalar potential $\phi(\mathbf{x})$ can be ascribed to bulk and surface densities of bound charge.

Consider a sphere of radius R consisting of a dielectric material with permittivity ϵ surrounded by a region of vacuum. A point-like electric charge q is placed at the centre of the sphere. Determine the density of bound charge on the surface of the sphere.

(b) Define the *magnetization* of a material and explain what is meant by the term *bound current*.

Consider a sample of material with spatially-dependent magnetization $\mathbf{M}(\mathbf{x})$ occupying a region V with surface S. Show that, in the absence of free currents, the resulting vector potential $\mathbf{A}(\mathbf{x})$ can be ascribed to bulk and surface densities of bound current.

Consider an infinite cylinder of radius r consisting of a material with permeability μ surrounded by a region of vacuum. A thin wire carrying current I is placed along the axis of the cylinder. Determine the direction and magnitude of the resulting bound current density on the surface of the cylinder. What is the magnetization $\mathbf{M}(\mathbf{x})$ on the surface of the cylinder?

37E General Relativity

(a) In the Newtonian weak-field limit, we can write the spacetime metric in the form

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)\delta_{ij}\, dx^i\, dx^j\,, \qquad (*)$$

where $\delta_{ij}dx^i dx^j = dx^2 + dy^2 + dz^2$ and the potential $\Phi(t,x,y,z)$, as well as the velocity v of particles moving in the gravitational field are assumed to be small, i.e.,

$$\Phi,\ \partial_t\Phi,\ \partial_{x^i}\Phi,\ v^2 \ll 1.$$

Use the geodesic equation for this metric to derive the equation of motion for a massive point particle in the Newtonian limit.

(b) The far-field limit of the Schwarzschild metric is a special case of $(*)$ given, in spherical coordinates, by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2)\,,$$

where now $M/r \ll 1$. For the following questions, state your results to first order in M/r, *i.e.* neglecting terms of $\mathcal{O}((M/r)^2)$.

(i) Let $r_1, r_2 \gg M$. Calculate the proper length S along the radial curve from r_1 to r_2 at fixed t, θ, φ.

(ii) Consider a massless particle moving radially from $r = r_1$ to $r = r_2$. According to an observer at rest at r_2, what time T elapses during this motion?

(iii) The *effective velocity* of the particle as seen by the observer at r_2 is defined as $v_{\text{eff}} := S/T$. Evaluate v_{eff} and then take the limit of this result as $r_1 \to r_2$. Briefly discuss the value of v_{eff} in this limit.

38C Fluid Dynamics II

A cylinder of radius a rotates about its axis with angular velocity Ω while its axis is fixed parallel to and at a distance $a + h_0$ from a rigid plane, where $h_0 \ll a$. Fluid of kinematic viscosity ν fills the space between the cylinder and the plane. Determine the gap width h between the cylinder and the plane as a function of a coordinate x parallel to the surface of the wall and orthogonal to the axis of the cylinder. What is the characteristic length scale, in the x direction, for changes in the gap width? Taking an appropriate approximation for $h(x)$, valid in the region where the gap width h is small, use lubrication theory to determine that the volume flux between the wall and the cylinder (per unit length along the axis) has magnitude $\frac{2}{3}a\Omega h_0$, and state its direction.

Evaluate the tangential shear stress τ on the surface of the cylinder. Approximating the torque on the cylinder (per unit length along the axis) in the form of an integral $T = a\int_{-\infty}^{\infty} \tau \, dx$, find the torque T to leading order in $h_0/a \ll 1$.

Explain the restriction $a^{1/2}\Omega h_0^{3/2}/\nu \ll 1$ for the theory to be valid.

[*You may use the facts that* $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$ *and* $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{8}$.]

39C Waves

A physical system permits one-dimensional wave propagation in the x-direction according to the equation

$$\left(1 - 2\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right)\frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial^4 \varphi}{\partial x^4} = 0\,.$$

Derive the corresponding dispersion relation and sketch graphs of frequency, phase velocity and group velocity as functions of the wavenumber. Waves of what wavenumber are at the front of a dispersing wave train arising from a localised initial disturbance? For waves of what wavenumbers do wave crests move faster or slower than a packet of waves?

Find the solution of the above equation for the initial disturbance given by

$$\varphi(x,0) = \int_{-\infty}^{\infty} 2A(k)e^{ikx}dk\,, \quad \frac{\partial \varphi}{\partial t}(x,0) = 0\,,$$

where $A^*(-k) = A(k)$, and A^* is the complex conjugate of A. Let $V = x/t$ be held fixed. Use the method of stationary phase to obtain a leading-order approximation to this solution for large t when $0 < V < V_m = (3\sqrt{3})/8$, where the solutions for the stationary points should be left in implicit form.

Very briefly discuss the nature of the solutions for $-V_m < V < 0$ and $|V| > V_m$.

[*Hint: You may quote the result that the large time behaviour of*

$$\Phi(x,t) = \int_{-\infty}^{\infty} A(k)e^{ikx - i\omega(k)t}dk\,,$$

due to a stationary point $k = \alpha$, is given by

$$\Phi(x,t) \sim \left(\frac{2\pi}{|\omega''(\alpha)|\,t}\right)^{\frac{1}{2}} A(\alpha)\, e^{i\alpha x - i\omega(\alpha)t + i\sigma\pi/4}\,,$$

where $\sigma = -\mathrm{sgn}(\omega''(\alpha))$.]

40E Numerical Analysis

The inverse discrete Fourier transform $\mathcal{F}_n^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is given by the formula

$$\boldsymbol{x} = \mathcal{F}_n^{-1}\boldsymbol{y}, \qquad \text{where} \qquad x_\ell = \sum_{j=0}^{n-1} \omega_n^{j\ell} y_j, \quad \ell = 0, \ldots, n-1.$$

Here, $\omega_n = \exp \frac{2\pi i}{n}$ is the primitive root of unity of degree n and $n = 2^p$, $p = 1, 2, \ldots$.

(a) Show how to assemble $\boldsymbol{x} = \mathcal{F}_{2m}^{-1}\boldsymbol{y}$ in a small number of operations if the Fourier transforms of the even and odd parts of $\boldsymbol{y}$,

$$\boldsymbol{x}^{(\mathrm{E})} = \mathcal{F}_m^{-1}\boldsymbol{y}^{(\mathrm{E})}, \qquad \boldsymbol{x}^{(\mathrm{O})} = \mathcal{F}_m^{-1}\boldsymbol{y}^{(\mathrm{O})},$$

are already known.

(b) Describe the Fast Fourier Transform (FFT) method for evaluating $\boldsymbol{x}$, and draw a diagram to illustrate the method for $n = 8$.

(c) Find the cost of the FFT method for $n = 2^p$ (only multiplications count).

(d) For $n = 4$ use the FFT method to find $\boldsymbol{x} = \mathcal{F}_n^{-1}\boldsymbol{y}$ when:

$$\text{(i)} \quad \boldsymbol{y} = (1, -1, 1, -1), \qquad \text{(ii)} \quad \boldsymbol{y} = (1, 1, -1, -1).$$

END OF PAPER