UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 7 June, 2018 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Groups, Rings and Modules

(a) Find all integer solutions to $x^2 + 5y^2 = 9$.

(b) Find all the irreducibles in $\mathbb{Z}[\sqrt{-5}]$ of norm 9.

2F Analysis II

For a continuous function $f = (f_1, f_2, \ldots, f_m) : [0,1] \to \mathbb{R}^m$, define

$$\int_0^1 f(t)\, dt = \left(\int_0^1 f_1(t)\, dt, \int_0^1 f_2(t)\, dt, \ldots, \int_0^1 f_m(t)\, dt \right).$$

Show that

$$\left\| \int_0^1 f(t)\, dt \right\|_2 \leqslant \int_0^1 \|f(t)\|_2\, dt$$

for every continuous function $f : [0,1] \to \mathbb{R}^m$, where $\| \cdot \|_2$ denotes the Euclidean norm on $\mathbb{R}^m$.

Find all continuous functions $f : [0,1] \to \mathbb{R}^m$ with the property that

$$\left\| \int_0^1 f(t)\, dt \right\| = \int_0^1 \|f(t)\|\, dt$$

regardless of the norm $\| \cdot \|$ on $\mathbb{R}^m$.

[*Hint: start by analysing the case when* $\|\cdot\|$ *is the Euclidean norm* $\|\cdot\|_2$.]

3E Metric and Topological Spaces

What does it mean to say that a topological space is *connected*? If X is a topological space and $x \in X$, show that there is a connected subspace K_x of X so that if S is any other connected subspace containing x then $S \subseteq K_x$.

Show that the sets K_x partition X.

4A Complex Methods

(a) Let $f(z) = (z^2-1)^{1/2}$. Define the branch cut of $f(z)$ as $[-1,1]$ such that

$$f(x) = +\sqrt{x^2-1} \qquad x > 1.$$

Show that $f(z)$ is an odd function.

(b) Let $g(z) = \left[(z-2)(z^2-1)\right]^{1/2}$.

(i) Show that $z=\infty$ is a branch point of $g(z)$.

(ii) Define the branch cuts of $g(z)$ as $[-1,1]\cup[2,\infty)$ such that

$$g(x) = e^{\pi i/2}\sqrt{|x-2||x^2-1|} \qquad x \in (1,2).$$

Find $g(0_\pm)$, where 0_+ denotes $z=0$ just above the branch cut, and 0_- denotes $z=0$ just below the branch cut.

5G Geometry

Consider a quadrilateral $ABCD$ in the hyperbolic plane whose sides are hyperbolic line segments. Suppose angles ABC, BCD and CDA are right-angles. Prove that AD is longer than BC.

[You may use without proof the distance formula in the upper-half-plane model

$$\rho(z_1,z_2) = 2\tanh^{-1}\left|\frac{z_1-z_2}{z_1-\bar{z}_2}\right| \quad .]$$

6B Variational Principles

For a particle of unit mass moving freely on a unit sphere, the Lagrangian in polar coordinates is

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\sin^2\theta\,\dot{\phi}^2\,.$$

Determine the equations of motion. Show that $l = \sin^2\theta\,\dot{\phi}$ is a conserved quantity, and use this result to simplify the equation of motion for θ. Deduce that

$$h = \dot{\theta}^2 + \frac{l^2}{\sin^2\theta}$$

is a conserved quantity. What is the interpretation of h?

7A Methods

(a) Determine the Green's function $G(x;\xi)$ satisfying

$$G'' - 4G' + 4G = \delta(x-\xi),$$

with $G(0;\xi) = G(1;\xi) = 0$. Here $'$ denotes differentiation with respect to x.

(b) Using the Green's function, solve

$$y'' - 4y' + 4y = e^{2x},$$

with $y(0) = y(1) = 0$.

8B Quantum Mechanics

What is meant by the statement that an operator is *Hermitian*?

Consider a particle of mass m in a real potential $V(x)$ in one dimension. Show that the Hamiltonian of the system is Hermitian.

Starting from the time-dependent Schrödinger equation, show that

$$\frac{d}{dt}\langle \hat{x} \rangle = \frac{1}{m}\langle \hat{p} \rangle, \qquad \frac{d}{dt}\langle \hat{p} \rangle = -\langle V'(\hat{x}) \rangle,$$

where $\hat{p}$ is the momentum operator and $\langle \hat{A} \rangle$ denotes the expectation value of the operator $\hat{A}$.

9H Markov Chains

The mathematics course at the University of Barchester is a three-year one. After the end-of-year examinations there are three possibilities:

(i) failing and leaving (probability p);

(ii) taking that year again (probability q);

(iii) going on to the next year (or graduating, if the current year is the third one) (probability r).

Thus there are five states for a student (1^{st} year, 2^{nd} year, 3^{rd} year, left without a degree, graduated).

Write down the 5×5 transition matrix. Classify the states, assuming $p, q, r \in (0,1)$. Find the probability that a student will eventually graduate.

SECTION II

10E Linear Algebra

State and prove the Cayley–Hamilton Theorem.

Let A be an $n \times n$ complex matrix. Using division of polynomials, show that if $p(x)$ is a polynomial then there is another polynomial $r(x)$ of degree at most $(n-1)$ such that $p(\lambda) = r(\lambda)$ for each eigenvalue λ of A and such that $p(A) = r(A)$.

Hence compute the $(1,1)$ entry of the matrix A^{1000} when

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}.$$

11G Groups, Rings and Modules

(a) State Gauss's Lemma.

(b) State and prove Eisenstein's criterion for the irreducibility of a polynomial.

(c) Determine whether or not the polynomial

$$f(X) = 2X^3 + 19X^2 - 54X + 3$$

is irreducible over $\mathbb{Q}$.

12F Analysis II

(a) Let $A \subset \mathbb{R}^m$ and let $f, f_n : A \to \mathbb{R}$ be functions for $n = 1, 2, 3, \ldots$. What does it mean to say that the sequence (f_n) *converges uniformly to* f *on* A? What does it mean to say that f is *uniformly continuous*?

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Determine whether each of the following statements is true or false. Give reasons for your answers.

(i) If $f_n(x) = f(x + 1/n)$ for each $n = 1, 2, 3, \ldots$ and each $x \in \mathbb{R}$, then $f_n \to f$ uniformly on $\mathbb{R}$.

(ii) If $g_n(x) = (f(x + 1/n))^2$ for each $n = 1, 2, 3, \ldots$ and each $x \in \mathbb{R}$, then $g_n \to (f)^2$ uniformly on $\mathbb{R}$.

(c) Let A be a closed, bounded subset of $\mathbb{R}^m$. For each $n = 1, 2, 3, \ldots$, let $g_n : A \to \mathbb{R}$ be a continuous function such that $(g_n(x))$ is a decreasing sequence for each $x \in A$. If $\delta \in \mathbb{R}$ is such that for each n there is $x_n \in A$ with $g_n(x_n) \geqslant \delta$, show that there is $x_0 \in A$ such that $\lim_{n\to\infty} g_n(x_0) \geqslant \delta$.

Deduce the following: If $f_n : A \to \mathbb{R}$ is a continuous function for each $n = 1, 2, 3, \ldots$ such that $(f_n(x))$ is a decreasing sequence for each $x \in A$, and if the pointwise limit of (f_n) is a continuous function $f : A \to \mathbb{R}$, then $f_n \to f$ uniformly on A.

13F Complex Analysis

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let $f : D \to \mathbb{C}$ be analytic.

(a) If there is a point $a \in D$ such that $|f(z)| \leqslant |f(a)|$ for all $z \in D$, prove that f is constant.

(b) If $f(0) = 0$ and $|f(z)| \leqslant 1$ for all $z \in D$, prove that $|f(z)| \leqslant |z|$ for all $z \in D$.

(c) Show that there is a constant C independent of f such that if $f(0) = 1$ and $f(z) \notin (-\infty, 0]$ for all $z \in D$ then $|f(z)| \leqslant C$ whenever $|z| \leqslant 1/2$.

[*Hint: you may find it useful to consider the principal branch of the map* $z \mapsto z^{1/2}$.]

(d) Does the conclusion in (c) hold if we replace the hypothesis $f(z) \notin (-\infty, 0]$ for $z \in D$ with the hypothesis $f(z) \neq 0$ for $z \in D$, and keep all other hypotheses? Justify your answer.

14G Geometry

Let U be an open subset of the plane $\mathbb{R}^2$, and let $\sigma : U \to S$ be a smooth parametrization of a surface S. A *coordinate curve* is an arc either of the form

$$\alpha_{v_0}(t) = \sigma(t, v_0)$$

for some constant v_0 and $t \in [u_1, u_2]$, or of the form

$$\beta_{u_0}(t) = \sigma(u_0, t)$$

for some constant u_0 and $t \in [v_1, v_2]$. A *coordinate rectangle* is a rectangle in S whose sides are coordinate curves.

Prove that all coordinate rectangles in S have opposite sides of the same length if and only if $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$ at all points of S, where E and G are the usual components of the first fundamental form, and (u, v) are coordinates in U.

15A Methods

Consider the Dirac delta function, $\delta(x)$, defined by the sampling property

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx = f(x_0),$$

for any suitable function $f(x)$ and real constant x_0.

(a) Show that $\delta(\alpha x) = |\alpha|^{-1}\delta(x)$ for any non-zero $\alpha \in \mathbb{R}$.

(b) Show that $x\delta'(x) = -\delta(x)$, where $'$ denotes differentiation with respect to x.

(c) Calculate

$$\int_{-\infty}^{\infty} f(x)\,\delta^{(m)}(x)\,dx,$$

where $\delta^{(m)}(x)$ is the m^{th} derivative of the delta function.

(d) For

$$\gamma_n(x) = \frac{1}{\pi}\frac{n}{(nx)^2+1},$$

show that $\lim_{n\to\infty} \gamma_n(x) = \delta(x)$.

(e) Find expressions in terms of the delta function and its derivatives for

(i)

$$\lim_{n\to\infty} n^3 x\, e^{-x^2 n^2}.$$

(ii)

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^n \cos(kx)\,dk.$$

(f) Hence deduce that

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-n}^{n} e^{ikx}\,dk = \delta(x).$$

[*You may assume that*

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi} \quad \textit{and} \quad \int_{-\infty}^{\infty} \frac{\sin y}{y}\,dy = \pi.]$$

16B Quantum Mechanics

What is the physical significance of the expectation value

$$\langle Q \rangle = \int \psi^*(x) Q \psi(x) dx$$

of an observable Q in the normalised state $\psi(x)$? Let P and Q be two observables. By considering the norm of $(Q + i\lambda P)\psi$ for real values of λ, show that

$$\langle Q^2 \rangle \langle P^2 \rangle \geqslant \frac{1}{4} |\langle [Q, P] \rangle|^2.$$

Deduce the generalised uncertainty relation

$$\Delta Q \; \Delta P \geqslant \frac{1}{2} |\langle [Q, P] \rangle|,$$

where the uncertainty ΔQ in the state $\psi(x)$ is defined by

$$(\Delta Q)^2 = \langle (Q - \langle Q \rangle)^2 \rangle.$$

A particle of mass m moves in one dimension under the influence of the potential $\frac{1}{2} m\omega^2 x^2$. By considering the commutator $[x, p]$, show that every energy eigenvalue E satisfies

$$E \geqslant \tfrac{1}{2} \hbar\omega.$$

17C Electromagnetism

Use Maxwell's equations to show that

$$\frac{d}{dt} \int_\Omega \left(\frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B} \right) dV + \int_\Omega \mathbf{J} \cdot \mathbf{E} \, dV = -\frac{1}{\mu_0} \int_{\partial\Omega} (\mathbf{E} \times \mathbf{B}) \cdot \mathbf{n} \, dS,$$

where $\Omega \subset \mathbb{R}^3$ is a bounded region, $\partial\Omega$ its boundary and $\mathbf{n}$ its outward–pointing normal. Give an interpretation for each of the terms in this equation.

A certain capacitor consists of two conducting, circular discs, each of large area A, separated by a small distance along their common axis. Initially, the plates carry charges q_0 and $-q_0$. At time $t = 0$ the plates are connected by a resistive wire, causing the charge on the plates to decay slowly as $q(t) = q_0 \, \mathrm{e}^{-\lambda t}$ for some constant λ. Construct the Poynting vector and show that energy flows radially out of the capacitor as it discharges.

18D Fluid Dynamics

A soap bubble of radius $a(t)$ is attached to the end of a long, narrow straw of internal radius ϵ and length L, the other end of which is open to the atmosphere. The pressure difference between the inside and outside of the bubble is $2\gamma/a$, where γ is the surface tension of the soap bubble. At time $t=0$, $a=a_0$ and the air in the straw is at rest. Assume that the flow of air through the straw is irrotational and consider the pressure drop along the straw to show that subsequently

$$a^3\ddot{a} + 2a^2\dot{a}^2 = -\frac{\gamma\epsilon^2}{2\rho L},$$

where ρ is the density of air.

By multiplying the equation by $2a\dot{a}$ and integrating, or otherwise, determine an implicit equation for $a(t)$ and show that the bubble disappears in a time

$$t = \frac{\pi}{2}\frac{a_0^2}{\epsilon}\left(\frac{\rho L}{2\gamma}\right)^{1/2}.$$

[*Hint: The substitution* $a = a_0 \sin\theta$ *can be used.*]

19D Numerical Analysis

Taylor's theorem for functions $f \in C^{k+1}[a,b]$ is given in the form

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^k}{k!}f^{(k)}(a) + R(x).$$

Use integration by parts to show that

$$R(x) = \frac{1}{k!}\int_a^x (x-\theta)^k f^{(k+1)}(\theta)\, d\theta.$$

Let λ_k be a linear functional on $C^{k+1}[a,b]$ such that $\lambda_k[p]=0$ for $p \in \mathbb{P}_k$. Show that

$$\lambda_k[f] = \frac{1}{k!}\int_a^b K(\theta) f^{(k+1)}(\theta)\, d\theta, \qquad (\dagger)$$

where the Peano kernel function $K(\theta) = \lambda_k\left[(x-\theta)_+^k\right]$. [*You may assume that the functional commutes with integration over a fixed interval.*]

The error in the mid-point rule for numerical quadrature on $[0,1]$ is given by

$$e[f] = \int_0^1 f(x)dx - f(\tfrac{1}{2}).$$

Show that $e[p]=0$ if p is a linear polynomial. Find the Peano kernel function corresponding to e explicitly and verify the formula $(\dagger)$ in the case $f(x)=x^2$.

20H Statistics

A treatment is suggested for a particular illness. The results of treating a number of patients chosen at random from those in a hospital suffering from the illness are shown in the following table, in which the entries a, b, c, d are numbers of patients.

	Recovery	Non-recovery
Untreated	a	b
Treated	c	d

Describe the use of Pearson's χ^2 statistic in testing whether the treatment affects recovery, and outline a justification derived from the generalised likelihood ratio statistic. Show that

$$\chi^2 = \frac{(ad-bc)^2(a+b+c+d)}{(a+b)(c+d)(a+c)(b+d)}.$$

[*Hint: You may find it helpful to observe that* $a(a+b+c+d)-(a+b)(a+c) = ad-bc$.]

Comment on the use of this statistical technique when

$$a = 50, \quad b = 10, \quad c = 15, \quad d = 5.$$

21H Optimisation

State and prove the Lagrangian Sufficiency Theorem.

The manufacturers, A and B, of two competing soap powders must plan how to allocate their advertising resources (X and Y pounds respectively) among n distinct geographical regions. If $x_i \geqslant 0$ and $y_i \geqslant 0$ denote, respectively, the resources allocated to area i by A and B then the number of packets sold by A and B in area i are

$$\frac{x_i u_i}{x_i + y_i}, \quad \frac{y_i u_i}{x_i + y_i}$$

respectively, where u_i is the total market in area i, and $u_1, u_2, \ldots, u_n$ are known constants. The difference between the amount sold by A and B is then

$$\sum_{i=1}^{n} \frac{x_i - y_i}{x_i + y_i} u_i.$$

A seeks to maximize this quantity, while B seeks to minimize it.

(i) If A knows B's allocation, how should A choose $x = (x_1, x_2, \ldots, x_n)$?

(ii) Determine the best strategies for A and B if each assumes the other will know its strategy and react optimally.

END OF PAPER