UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 8 June, 2017 9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *…,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Explain what is meant by an *Euler pseudoprime* and a *strong pseudoprime*. Show that 65 is an Euler pseudoprime to the base b if and only if $b^2 \equiv \pm 1 \pmod{65}$. How many such bases are there? Show that the bases for which 65 is a strong pseudoprime do *not* form a subgroup of $(\mathbb{Z}/65\mathbb{Z})^\times$.

2F Topics In Analysis

(a) Suppose that $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function such that there exists a $K > 0$ with $\|g(\mathbf{x}) - \mathbf{x}\| \leqslant K$ for all $\mathbf{x} \in \mathbb{R}^2$. By constructing a suitable map f from the closed unit disc into itself, show that there exists a $\mathbf{t} \in \mathbb{R}^2$ with $g(\mathbf{t}) = \mathbf{0}$.

(b) Show that g is surjective.

(c) Show that the result of part (b) may be false if we drop the condition that g is continuous.

3G Coding & Cryptography

Find and describe all binary cyclic codes of length 7. Pair each code with its dual code. Justify your answer.

4H Automata and Formal Languages

(a) Define what it means for a context-free grammar (CFG) to be in *Chomsky normal form* (CNF). Give an example, with justification, of a context-free language (CFL) which is not defined by any CFG in CNF.

(b) Show that the intersection of two CFLs need not be a CFL.

(c) Let L be a CFL over an alphabet Σ. Show that $\Sigma^* \setminus L$ need not be a CFL.

5J Statistical Modelling

For Fisher's method of Iteratively Reweighted Least-Squares and Newton–Raphson optimisation of the log-likelihood, the vector of parameters β is updated using an iteration

$$\beta^{(m+1)} = \beta^{(m)} + M(\beta^{(m)})^{-1} U(\beta^{(m)}) \,,$$

for a specific function M. How is M defined in each method?

Prove that they are identical in a Generalised Linear Model with the canonical link function.

6B Mathematical Biology

A stochastic birth-death process has a master equation given by

$$\frac{dp(n,t)}{dt} = \lambda \left[p(n-1,t) - p(n,t) \right] + \beta \left[(n+1)\, p(n+1,t) - n\, p(n,t) \right] \,,$$

where $p(n,t)$ is the probability that there are n individuals in the population at time t for $n = 0, 1, 2, \ldots$ and $p(n,t) = 0$ for $n < 0$.

Give the corresponding Fokker–Planck equation for this system.

Use this Fokker–Planck equation to find expressions for $\frac{d}{dt}\langle x \rangle$ and $\frac{d}{dt}\langle x^2 \rangle$.

[*Hint: The general form for a Fokker–Planck equation in $P(x,t)$ is*

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(AP) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(BP) \,.$$

You may use this general form, stating how $A(x)$ and $B(x)$ are constructed. Alternatively, you may derive a Fokker–Plank equation directly by working from the master equation.]

7E Further Complex Methods

Find all the singular points of the differential equation

$$z\frac{d^2 y}{dz^2} + (2-z)\frac{dy}{dz} - y = 0$$

and determine whether they are regular or irregular singular points.

By writing $y(z) = f(z)/z$, find two linearly independent solutions to this equation.

Comment on the relationship of your solutions to the nature of the singular points of the original differential equation.

8E Classical Dynamics

Define an *integrable system* with $2n$-dimensional phase space. Define *angle-action variables.*

Consider a two-dimensional phase space with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}q^{2k},$$

where k is a positive integer and the mass $m = m(t)$ changes slowly in time. Use the fact that the action is an adiabatic invariant to show that the energy varies in time as m^c, where c is a constant which should be found.

9C Cosmology

(a) In the early universe electrons, protons and neutral hydrogen are in thermal equilibrium and interact via,

$$e^- + p^+ \leftrightharpoons H + \gamma\,.$$

The non-relativistic number density of particles in thermal equlibrium is

$$n_i = g_i \left(\frac{2\pi m_i kT}{h^2}\right)^{\frac{3}{2}} \exp\left(\frac{\mu_i - m_i c^2}{kT}\right)\,,$$

where, for each species i, g_i is the number of degrees of freedom, m_i is its mass, and μ_i is its chemical potential. [You may assume $g_e = g_p = 2$ and $g_H = 4$.]

Stating any assumptions required, use these expressions to derive the Saha equation which governs the relative abundances of electrons, protons and hydrogen,

$$\frac{n_e n_p}{n_H} = \left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}} \exp\left(-\frac{I}{kT}\right)\,,$$

where I is the binding energy of hydrogen, which should be defined.

(b) Naively, we might expect that the majority of electrons and protons combine to form neutral hydrogen once the temperature drops below the binding energy, i.e. $kT \lesssim I$. In fact recombination does not happen until a much lower temperature, when $kT \approx 0.03I$. Briefly explain why this is.

[*Hint: It may help to consider the relative abundances of particles in the early universe.*]

SECTION II

10G Number Theory

Let d be a positive integer which is not a square. Assume that the continued fraction expansion of $\sqrt{d}$ takes the form $[a_0, \overline{a_1, a_2, \ldots, a_m}]$.

(a) Define the *convergents* p_n/q_n, and show that p_n and q_n are coprime.

(b) The complete quotients θ_n may be written in the form $(\sqrt{d}+r_n)/s_n$, where r_n and s_n are rational numbers. Use the relation

$$\sqrt{d} = \frac{\theta_n p_{n-1} + p_{n-2}}{\theta_n q_{n-1} + q_{n-2}}$$

to find formulae for r_n and s_n in terms of the p's and q's. Deduce that r_n and s_n are integers.

(c) Prove that Pell's equation $x^2 - dy^2 = 1$ has infinitely many solutions in integers x and y.

(d) Find integers x and y satisfying $x^2 - 67y^2 = -2$.

11H Automata and formal languages

(a) Given $A, B \subseteq \mathbb{N}$, define a *many-one reduction* of A to B. Show that if B is recursively enumerable (r.e.) and $A \leqslant_m B$ then A is also recursively enumerable.

(b) State the *s-m-n* theorem, and use it to prove that a set $X \subseteq \mathbb{N}$ is r.e. if and only if $X \leqslant_m \mathbb{K}$.

(c) Consider the sets of integers $P, Q \subseteq \mathbb{N}$ defined via

$$P := \{n \in \mathbb{N} \mid n \text{ codes a program and } W_n \text{ is finite}\}$$
$$Q := \{n \in \mathbb{N} \mid n \text{ codes a program and } W_n \text{ is recursive}\}.$$

Show that $P \leqslant_m Q$.

12B Mathematical Biology

In a discrete-time model, adults and larvae of a population at time n are represented by a_n and b_n respectively. The model is represented by the equations

$$\begin{aligned} a_{n+1} &= (1-k)a_n + \frac{b_n}{1+a_n}\,, \\ b_{n+1} &= \mu\, a_n\,. \end{aligned}$$

You may assume that $k \in (0,1)$ and $\mu > 0$. Give an explanation of what each of the terms represents, and hence give a description of the population model.

By combining the equations to describe the dynamics purely in terms of the adults, find all equilibria of the system. Show that the equilibrium with the population absent ($a = 0$) is unstable exactly when there exists an equilibrium with the population present ($a > 0$).

Give the condition on μ and k for the equilibrium with $a > 0$ to be stable, and sketch the corresponding region in the (k, μ) plane.

What happens to the population close to the boundaries of this region?

If this model was modified to include stochastic effects, briefly describe qualitatively the likely dynamics near the boundaries of the region found above.

13C Cosmology

(a) The scalar moment of inertia for a system of N particles is given by

$$I = \sum_{i=1}^{N} m_i \, \mathbf{r}_i \cdot \mathbf{r}_i \, ,$$

where m_i is the particle's mass and $\mathbf{r}_i$ is a vector giving the particle's position. Show that, for non-relativistic particles,

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i$$

where K is the total kinetic energy of the system and $\mathbf{F}_i$ is the total force on particle i.

Assume that any two particles i and j interact gravitationally with potential energy

$$V_{ij} = -\frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} \, .$$

Show that

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{r}_i = V \, ,$$

where V is the total potential energy of the system. Use the above to prove the virial theorem.

(b) Consider an approximately spherical overdensity of stationary non-interacting massive particles with initial constant density ρ_i and initial radius R_i. Assuming the system evolves until it reaches a stable virial equilibrium, what will the final ρ and R be in terms of their initial values? Would this virial solution be stable if our particles were baryonic rather than non-interacting? Explain your answer.

14H Logic and Set Theory

State and prove Zorn's Lemma. [You may assume Hartogs' Lemma.] Indicate clearly where in your proof you have made use of the Axiom of Choice.

Show that $\mathbb{R}$ has a basis as a vector space over $\mathbb{Q}$.

Let V be a vector space over $\mathbb{Q}$. Show that all bases of V have the same cardinality.

[*Hint: How does the cardinality of* V *relate to the cardinality of a given basis?*]

15H Graph Theory

Define the *Ramsey numbers* $R(s,t)$ for integers $s,t \geqslant 2$. Show that $R(s,t)$ exists for all $s,t \geqslant 2$. Show also that $R(s,s) \leqslant 4^s$ for all $s \geqslant 2$.

Let $t \geqslant 2$ be fixed. Give a red-blue colouring of the edges of K_{2t-2} for which there is no red K_t and no blue odd cycle. Show, however, that for any red-blue colouring of the edges of K_{2t-1} there must exist either a red K_t or a blue odd cycle.

16I Galois Theory

(a) Let F be a finite field of characteristic p. Show that F is a finite Galois extension of the field F_p of p elements, and that the Galois group of F over F_p is cyclic.

(b) Find the Galois groups of the following polynomials:

(i) $t^4 + 1$ over F_3.

(ii) $t^3 - t - 2$ over F_5.

(iii) $t^4 - 1$ over F_7.

17G Representation Theory

(a) State Burnside's $p^a q^b$ theorem.

(b) Let P be a non-trivial group of prime power order. Show that if H is a non-trivial normal subgroup of P, then $H \cap Z(P) \neq \{1\}$.

Deduce that a non-abelian simple group cannot have an abelian subgroup of prime power index.

(c) Let ρ be a representation of the finite group G over $\mathbb{C}$. Show that $\delta : g \mapsto \det(\rho(g))$ is a linear character of G. Assume that $\delta(g) = -1$ for some $g \in G$. Show that G has a normal subgroup of index 2.

Now let E be a group of order $2k$, where k is an odd integer. By considering the regular representation of E, or otherwise, show that E has a normal subgroup of index 2.

Deduce that if H is a non-abelian simple group of order less than 80, then H has order 60.

18I Algebraic Topology

The n-torus is the product of n circles:

$$T^n = \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}} .$$

For all $n \geqslant 1$ and $0 \leqslant k \leqslant n$, compute $H_k(T^n)$.

[You may assume that relevant spaces are triangulable, but you should state carefully any version of any theorem that you use.]

19F Linear Analysis

Let K be a non-empty compact Hausdorff space and let $C(K)$ be the space of real-valued continuous functions on K.

(i) State the real version of the Stone–Weierstrass theorem.

(ii) Let A be a closed subalgebra of $C(K)$. Prove that $f \in A$ and $g \in A$ implies that $m \in A$ where the function $m : K \to \mathbb{R}$ is defined by $m(x) = \max\{f(x), g(x)\}$. [You may use without proof that $f \in A$ implies $|f| \in A$.]

(iii) Prove that K is normal and state Urysohn's Lemma.

(iv) For any $x \in K$, define $\delta_x \in C(K)^*$ by $\delta_x(f) = f(x)$ for $f \in C(K)$. Justifying your answer carefully, find

$$\inf_{x \neq y} \|\delta_x - \delta_y\|.$$

20F Analysis of Functions

Denote by $C_0(\mathbb{R}^n)$ the space of continuous complex-valued functions on $\mathbb{R}^n$ converging to zero at infinity. Denote by $\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} f(x)\,dx$ the Fourier transform of $f \in L^1(\mathbb{R}^n)$.

(i) Prove that the image of $L^1(\mathbb{R}^n)$ under $\mathcal{F}$ is included and dense in $C_0(\mathbb{R}^n)$, and that $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ is injective. [Fourier inversion can be used without proof when properly stated.]

(ii) Calculate the Fourier transform of $\chi_{[a,b]}$, the characteristic function of $[a,b] \subset \mathbb{R}$.

(iii) Prove that $g_n := \chi_{[-n,n]} * \chi_{[-1,1]}$ belongs to $C_0(\mathbb{R})$ and is the Fourier transform of a function $h_n \in L^1(\mathbb{R})$, which you should determine.

(iv) Using the functions h_n, g_n and the open mapping theorem, deduce that the Fourier transform is not surjective from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$.

21F Riemann Surfaces

Let $n \geqslant 2$ be a positive even integer. Consider the subspace R of $\mathbb{C}^2$ given by the equation $w^2 = z^n - 1$, where (z, w) are coordinates in $\mathbb{C}^2$, and let $\pi : R \to \mathbb{C}$ be the restriction of the projection map to the first factor. Show that R has the structure of a Riemann surface in such a way that π becomes an analytic map. If τ denotes projection onto the second factor, show that τ is also analytic. [You may assume that R is connected.]

Find the ramification points and the branch points of both π and τ. Compute the ramification indices at the ramification points.

Assume that, by adding finitely many points, it is possible to compactify R to a Riemann surface $\overline{R}$ such that π extends to an analytic map $\overline{\pi} : \overline{R} \to \mathbb{C}_\infty$. Find the genus of $\overline{R}$ (as a function of n).

22I Algebraic Geometry

(a) Define what it means to give a *rational map* between algebraic varieties. Define a *birational map.*

(b) Let
$$X = Z(y^2 - x^2(x-1)) \subseteq \mathbb{A}^2.$$
Define a birational map from X to $\mathbb{A}^1$. [*Hint: Consider lines through the origin.*]

(c) Let $Y \subseteq \mathbb{A}^3$ be the surface given by the equation
$$x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1 = 0.$$
Consider the blow-up $X \subseteq \mathbb{A}^3 \times \mathbb{P}^2$ of $\mathbb{A}^3$ at the origin, i.e. the subvariety of $\mathbb{A}^3 \times \mathbb{P}^2$ defined by the equations $x_i y_j = x_j y_i$ for $1 \leqslant i < j \leqslant 3$, with y_1, y_2, y_3 coordinates on $\mathbb{P}^2$. Let $\varphi : X \to \mathbb{A}^3$ be the projection and $E = \varphi^{-1}(0)$. Recall that the proper transform $\widetilde{Y}$ of Y is the closure of $\varphi^{-1}(Y) \setminus E$ in X. Give equations for $\widetilde{Y}$, and describe the fibres of the morphism $\varphi|_{\widetilde{Y}} : \widetilde{Y} \to Y$.

23I Differential Geometry

Let $S \subset \mathbb{R}^N$ be a manifold and let $\alpha : [a, b] \to S \subset \mathbb{R}^N$ be a smooth regular curve on S. Define the *total length* $L(\alpha)$ and the *arc length* parameter s. Show that α can be reparametrized by arc length.

Let $S \subset \mathbb{R}^3$ denote a regular surface, let $p, q \in S$ be distinct points and let $\alpha : [a, b] \to S$ be a smooth regular curve such that $\alpha(a) = p$, $\alpha(b) = q$. We say that α is *length minimising* if for all smooth regular curves $\tilde{\alpha} : [a, b] \to S$ with $\tilde{\alpha}(a) = p$, $\tilde{\alpha}(b) = q$, we have $L(\tilde{\alpha}) \geqslant L(\alpha)$. By deriving a formula for the derivative of the energy functional corresponding to a variation of α, show that a length minimising curve is necessarily a geodesic. [You may use the following fact: given a smooth vector field $V(t)$ along α with $V(a) = V(b) = 0$, there exists a variation $\alpha(s, t)$ of α such that $\partial_s \alpha(s, t)|_{s=0} = V(t)$.]

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit sphere and let S denote the surface $\mathbb{S}^2 \setminus (0, 0, 1)$. For which pairs of points $p, q \in S$ does there exist a length minimising smooth regular curve $\alpha : [a, b] \to S$ with $\alpha(a) = p$ and $\alpha(b) = q$? Justify your answer.

24J Probability and Measure

(a) Suppose that $\mathcal{X} = (X_n)$ is a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Give the definition of what it means for $\mathcal{X}$ to be *uniformly integrable*.

(b) State and prove Hölder's inequality.

(c) Explain what it means for a family of random variables to be L^p *bounded*. Prove that an L^p bounded sequence is uniformly integrable provided $p > 1$.

(d) Prove or disprove: every sequence which is L^1 bounded is uniformly integrable.

25K Applied Probability

(a) Define the *Moran model* and *Kingman's n-coalescent*. Define *Kingman's infinite coalescent*.

Show that Kingman's infinite coalescent comes down from infinity. In other words, with probability one, the number of blocks of Π_t is finite at any time $t > 0$.

(b) Give the definition of a *renewal process*.

Let (X_i) denote the sequence of inter-arrival times of the renewal process N. Suppose that $\mathbb{E}[X_1] > 0$.

Prove that $\mathbb{P}(N(t) \to \infty \text{ as } t \to \infty) = 1$.

Prove that $\mathbb{E}\big[e^{\theta N(t)}\big] < \infty$ for some strictly positive θ.

[*Hint: Consider the renewal process with inter-arrival times* $X'_k = \varepsilon \mathbf{1}(X_k \geqslant \varepsilon)$ *for some suitable* $\varepsilon > 0$.]

26K Principles of Statistics

We consider the problem of estimating an unknown θ_0 in a statistical model $\{f(x,\theta)\,,\, \theta \in \Theta\}$ where $\Theta \subset \mathbb{R}$, based on n i.i.d. observations $X_1, \ldots, X_n$ whose distribution has p.d.f. $f(x, \theta_0)$.

In all the parts below you may assume that the model satisfies necessary regularity conditions.

(a) Define the *score function* S_n of θ. Prove that $S_n(\theta_0)$ has mean 0.

(b) Define the *Fisher Information* $I(\theta)$. Show that it can also be expressed as

$$I(\theta) = -\mathbb{E}_\theta\Big[\frac{d^2}{d\theta^2} \log f(X_1, \theta)\Big]\,.$$

(c) Define the *maximum likelihood estimator* $\hat\theta_n$ of θ. Give without proof the limits of $\hat\theta_n$ and of $\sqrt{n}(\hat\theta_n - \theta_0)$ (in a manner which you should specify). [Be as precise as possible when describing a distribution.]

(d) Let $\psi : \Theta \to \mathbb{R}$ be a continuously differentiable function, and $\tilde\theta_n$ another estimator of θ_0 such that $|\hat\theta_n - \tilde\theta_n| \leqslant 1/n$ with probability 1. Give the limits of $\psi(\tilde\theta_n)$ and of $\sqrt{n}(\psi(\tilde\theta_n) - \psi(\theta_0))$ (in a manner which you should specify).

27J Stochastic Financial Models

(a) State the fundamental theorem of asset pricing for a multi-period model.

Consider a market model in which there is no arbitrage, the prices for all European put and call options are already known and there is a riskless asset $S^0 = (S^0_t)_{t\in\{0,\ldots,T\}}$ with $S^0_t = (1+r)^t$ for some $r \geqslant 0$. The holder of a so-called 'chooser option' $C(K,t_0,T)$ has the right to choose at a preassigned time $t_0 \in \{0,1,\ldots,T\}$ between a European call and a European put option on the same asset S^1, both with the same strike price K and the same maturity T. [We assume that at time t_0 the holder will take the option having the higher price at that time.]

(b) Show that the payoff function of the chooser option is given by

$$C(K,t_0,T) = \begin{cases} (S^1_T - K)^+ & \text{if } S^1_{t_0} > K(1+r)^{t_0-T}, \\ (K - S^1_T)^+ & \text{otherwise.} \end{cases}$$

(c) Show that the price $\pi(C(K,t_0,T))$ of the chooser option $C(K,t_0,T)$ is given by

$$\pi(C(K,t_0,T)) = \pi\big(EC(K,T)\big) + \pi\big(EP\big(K(1+r)^{t_0-T}, t_0\big)\big),$$

where $\pi\big(EC(K,T)\big)$ and $\pi\big(EP(K,T)\big)$ denote the price of a European call and put option, respectively, with strike K and maturity T.

28K Optimization and Control

A particle follows a discrete-time trajectory on $\mathbb{R}$ given by

$$x_{t+1} = (Ax_t + u_t)\xi_t + \epsilon_t$$

for $t = 1, 2, \ldots, T$. Here $T \geqslant 2$ is a fixed integer, A is a real constant, x_t and u_t are the position of the particle and control action at time t, respectively, and $(\xi_t, \epsilon_t)_{t=1}^T$ is a sequence of independent random vectors with

$$\mathbb{E}\,\xi_t = \mathbb{E}\,\epsilon_t = 0,\ \operatorname{var}(\xi_t) = V_\xi > 0,\ \operatorname{var}(\epsilon_t) = V_\epsilon > 0 \text{ and } \operatorname{cov}(\xi_t, \epsilon_t) = 0\,.$$

Find the optimal control, i.e. the control action u_t, defined as a function of $(x_1, \ldots, x_t; u_1, \ldots, u_{t-1})$, that minimizes

$$\sum_{t=1}^{T} x_t^2 + c\sum_{t=1}^{T-1} u_t^2\,,$$

where $c > 0$ is given.

On which of V_ϵ and V_ξ does the optimal control depend?

Find the limiting form of the optimal control as $T \to \infty$, and the minimal average cost per unit time.

29E Asymptotic Methods

Consider the integral representation for the modified Bessel function

$$I_0(x) = \frac{1}{2\pi i}\oint_C t^{-1}\exp\left[\frac{ix}{2}\left(t - \frac{1}{t}\right)\right]\,dt,$$

where C is a simple closed contour containing the origin, taken anti-clockwise.

Use the method of steepest descent to determine the full asymptotic expansion of $I_0(x)$ for large real positive x.

30A Dynamical Systems

State, without proof, the centre manifold theorem. Show that the fixed point at the origin of the system

$$\begin{aligned} \dot{x} &= y - x + ax^3 , \\ \dot{y} &= rx - y - yz , \\ \dot{z} &= xy - z , \end{aligned}$$

where $a \neq 1$ is a constant, is nonhyperbolic at $r = 1$. What are the dimensions of the linear stable and (non-extended) centre subspaces at this point?

Make the substitutions $2u = x+y$, $2v = x-y$ and $\mu = r-1$ and derive the resultant equations for $\dot{u}, \dot{v}$ and $\dot{z}$.

The extended centre manifold is given by

$$v = V(u,\mu), \qquad z = Z(u,\mu) ,$$

where V and Z can be expanded as power series about $u = \mu = 0$. What is known about V and Z from the centre manifold theorem? Assuming that $\mu = O(u^2)$, determine Z to $O(u^2)$ and V to $O(u^3)$. Hence obtain the evolution equation on the centre manifold correct to $O(u^3)$, and identify the type of bifurcation distinguishing between the cases $a > 1$ and $a < 1$.

If now $a = 1$, assume that $\mu = O(u^4)$ and extend your calculations of Z to $O(u^4)$ and of the dynamics on the centre manifold to $O(u^5)$. Hence sketch the bifurcation diagram in the neighbourhood of $u = \mu = 0$.

31A Integrable Systems

Let $u = u(x,t)$ be a smooth solution to the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0$$

which decays rapidly as $|x| \to \infty$ and let $L = -\partial_x^2 + u$ be the associated Schrödinger operator. You may assume L and $A = 4\partial_x^3 - 3(u\partial_x + \partial_x u)$ constitute a Lax pair for KdV.

Consider a solution to $L\varphi = k^2\varphi$ which has the asymptotic form

$$\varphi(x,k,t) = \begin{cases} e^{-\mathrm{i}kx}, & \text{as } x \to -\infty, \\ a(k,t)e^{-\mathrm{i}kx} + b(k,t)e^{\mathrm{i}kx}, & \text{as } x \to +\infty. \end{cases}$$

Find evolution equations for a and b. Deduce that $a(k,t)$ is t-independent.

By writing φ in the form

$$\varphi(x,k,t) = \exp\left[-\mathrm{i}kx + \int_{-\infty}^{x} S(y,k,t)\,\mathrm{d}y\right], \quad S(x,k,t) = \sum_{n=1}^{\infty} \frac{S_n(x,t)}{(2\mathrm{i}k)^n},$$

show that

$$a(k,t) = \exp\left[\int_{-\infty}^{\infty} S(x,k,t)\,\mathrm{d}x\right].$$

Deduce that $\{\int_{-\infty}^{\infty} S_n(x,t)\,\mathrm{d}x\}_{n=1}^{\infty}$ are first integrals of KdV.

By writing a differential equation for $S = X + \mathrm{i}Y$ (with X, Y real), show that these first integrals are trivial when n is even.

32C Principles of Quantum Mechanics

The angular momentum operators $\mathbf{J} = (J_1, J_2, J_3)$ obey the commutation relations

$$[J_3, J_\pm] = \pm J_\pm\,,$$

$$[J_+, J_-] = 2J_3\,,$$

where $J_\pm = J_1 \pm iJ_2$.

A quantum mechanical system involves the operators $a, a^\dagger, b$ and $b^\dagger$ such that

$$[a, a^\dagger] = [b, b^\dagger] = 1\,,$$

$$[a, b] = [a^\dagger, b] = [a, b^\dagger] = [a^\dagger, b^\dagger] = 0.$$

Define $K_+ = a^\dagger b$, $K_- = ab^\dagger$ and $K_3 = \frac{1}{2}(a^\dagger a - b^\dagger b)$. Show that $K_\pm$ and K_3 obey the same commutation relations as $J_\pm$ and J_3.

Suppose that the system is in the state $|0\rangle$ such that $a|0\rangle = b|0\rangle = 0$. Show that $(a^\dagger)^2|0\rangle$ is an eigenstate of K_3. Let $K^2 = \frac{1}{2}(K_+K_- + K_-K_+) + K_3^2$. Show that $(a^\dagger)^2|0\rangle$ is an eigenstate of K^2 and find the eigenvalue. How many other states do you expect to find with same value of K^2? Find them.

33C Applications of Quantum Mechanics

A particle of mass m and charge q moving in a uniform magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = (0, 0, B)$ is described by the Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2$$

where $\mathbf{p}$ is the canonical momentum, which obeys $[x_i, p_j] = i\hbar\delta_{ij}$. The *mechanical momentum* is defined as $\boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}$. Show that

$$[\pi_x, \pi_y] = iq\hbar B\,.$$

Define

$$a = \frac{1}{\sqrt{2q\hbar B}}(\pi_x + i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2q\hbar B}}(\pi_x - i\pi_y)\,.$$

Derive the commutation relation obeyed by a and $a^\dagger$. Write the Hamiltonian in terms of a and $a^\dagger$ and hence solve for the spectrum.

In symmetric gauge, states in the lowest Landau level with $k_z = 0$ have wavefunctions

$$\psi(x, y) = (x + iy)^M\, e^{-qBr^2/4\hbar}$$

where $r^2 = x^2 + y^2$ and M is a positive integer. By considering the profiles of these wavefunctions, estimate how many lowest Landau level states can fit in a disc of radius R.

34D Statistical Physics

(a) Describe the *Carnot cycle* using plots in the (p, V)-plane and the (T, S)-plane. In which steps of the cycle is heat absorbed or emitted by the gas? In which steps is work done on, or by, the gas?

(b) An ideal monatomic gas undergoes a reversible cycle described by a triangle in the (p, V)-plane with vertices at the points A, B, C with coordinates (p_0, V_0), $(2p_0, V_0)$ and $(p_0, 2V_0)$ respectively. The cycle is traversed in the order $ABCA$.

(i) Write down the equation of state and an expression for the internal energy of the gas.

(ii) Derive an expression relating TdS to dp and dV. Use your expression to calculate the heat supplied to, or emitted by, the gas along AB and CA.

(iii) Show that heat is supplied to the gas along part of the line BC, and is emitted by the gas along the other part of the line.

(iv) Calculate the efficiency $\eta = W/Q$ where W is the total work done by the cycle and Q is the total heat supplied.

35D Electrodynamics

By considering the force per unit volume $\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$ on a charge density ρ and current density $\mathbf{J}$ due to an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, show that

$$\frac{\partial g_i}{\partial t} + \frac{\partial \sigma_{ij}}{\partial x_j} = -f_i \,,$$

where $\mathbf{g} = \epsilon_0 \mathbf{E} \times \mathbf{B}$ and the symmetric tensor σ_{ij} should be specified.

Give the physical interpretation of $\mathbf{g}$ and σ_{ij} and explain how σ_{ij} can be used to calculate the net electromagnetic force exerted on the charges and currents within some region of space in static situations.

The plane $x = 0$ carries a uniform charge σ per unit area and a current K per unit length along the z-direction. The plane $x = d$ carries the opposite charge and current. Show that between these planes

$$\sigma_{ij} = \frac{\sigma^2}{2\epsilon_0} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\mu_0 K^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad (*)$$

and $\sigma_{ij} = 0$ for $x < 0$ and $x > d$.

Use $(*)$ to find the electromagnetic force per unit area exerted on the charges and currents in the $x = 0$ plane. Show that your result agrees with direct calculation of the force per unit area based on the Lorentz force law.

If the current K is due to the motion of the charge σ with speed v, is it possible for the force between the planes to be repulsive?

36D General Relativity

Let $\mathcal{M}$ be a two-dimensional manifold with metric $\boldsymbol{g}$ of signature $-+$.

(i) Let $p \in \mathcal{M}$. Use normal coordinates at the point p to show that one can choose two null vectors $\mathbf{V}$, $\mathbf{W}$ that form a basis of the vector space $\mathcal{T}_p(\mathcal{M})$.

(ii) Consider the interval $I \subset \mathbb{R}$. Let $\gamma : I \to \mathcal{M}$ be a null curve through p and $\mathbf{U} \neq 0$ be the tangent vector to γ at p. Show that the vector $\mathbf{U}$ is either parallel to $\mathbf{V}$ or parallel to $\mathbf{W}$.

(iii) Show that every null curve in $\mathcal{M}$ is a null geodesic.

[*Hint: You may wish to consider the acceleration* $a^\alpha = U^\beta \nabla_\beta U^\alpha$.]

(iv) By providing an example, show that not every null curve in four-dimensional Minkowski spacetime is a null geodesic.

37B Fluid Dynamics II

A spherical bubble of radius a moves with velocity $\mathbf{U}$ through a viscous fluid that is at rest far from the bubble. The pressure and velocity fields outside the bubble are given by

$$p = \mu \frac{a}{r^3} \mathbf{U} \cdot \mathbf{x} \quad \text{and} \quad \mathbf{u} = \frac{a}{2r}\mathbf{U} + \frac{a}{2r^3}(\mathbf{U} \cdot \mathbf{x})\mathbf{x}\,,$$

respectively, where μ is the dynamic viscosity of the fluid, $\mathbf{x}$ is the position vector from the centre of the bubble and $r = |\mathbf{x}|$. Using suffix notation, or otherwise, show that these fields satisfy the Stokes equations.

Obtain an expression for the stress tensor for the fluid outside the bubble and show that the velocity field above also satisfies all the appropriate boundary conditions.

Compute the drag force on the bubble.

[*Hint: You may use*

$$\int_S n_i n_j \, dS = \frac{4}{3}\pi a^2 \delta_{ij},$$

where the integral is taken over the surface of a sphere of radius a and $\mathbf{n}$ *is the outward unit normal to the surface.*]

38B Waves

Waves propagating in a slowly-varying medium satisfy the local dispersion relation $\omega = \Omega(\mathbf{k}; \mathbf{x}, t)$ in the standard notation. Derive the ray-tracing equations

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i}, \quad \frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}, \quad \frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t}$$

governing the evolution of a wave packet specified by $\varphi(\mathbf{x}, t) = A(\mathbf{x}, t; \varepsilon)e^{i\theta(\mathbf{x},t)/\varepsilon}$, where $0 < \varepsilon \ll 1$. A formal justification is not required, but the meaning of the d/dt notation should be carefully explained.

The dispersion relation for two-dimensional, small amplitude, internal waves of wavenumber $\mathbf{k} = (k, 0, m)$, relative to Cartesian coordinates (x, y, z) with z vertical, propagating in an inviscid, incompressible, stratified fluid that would otherwise be at rest, is given by

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2},$$

where N is the Brunt–Väisälä frequency and where you may assume that $k > 0$ and $\omega > 0$. Derive the modified dispersion relation if the fluid is not at rest, and instead has a slowly-varying mean flow $(U(z), 0, 0)$.

In the case that $U'(z) > 0, U(0) = 0$ and N is constant, show that a disturbance with wavenumber $\mathbf{k} = (k, 0, 0)$ generated at $z = 0$ will propagate upwards but cannot go higher than a critical level $z = z_c$, where $U(z_c)$ is equal to the apparent wave speed in the x-direction. Find expressions for the vertical wave number m as $z \to z_c$ from below, and show that it takes an infinite time for the wave to reach the critical level.

39A Numerical Analysis

Let A be a real symmetric $n \times n$ matrix with real and distinct eigenvalues $0 = \lambda_1 < \cdots < \lambda_{n-1} = 1 < \lambda_n$ and a corresponding orthogonal basis of normalized real eigenvectors $(\mathbf{w}_i)_{i=1}^n$.

To estimate the eigenvector $\mathbf{w}_n$ of A whose eigenvalue is λ_n, the power method with shifts is employed which has the following form:

$$\mathbf{y} = (A - s_k I)\mathbf{x}^{(k)}, \qquad \mathbf{x}^{(k+1)} = \mathbf{y}/\|\mathbf{y}\|, \qquad s_k \in \mathbb{R}, \qquad k = 0, 1, 2, \ldots$$

Three versions of this method are considered:

(i) no shift: $s_k \equiv 0$;

(ii) single shift: $s_k \equiv \frac{1}{2}$;

(iii) double shift: $s_{2\ell} \equiv s_0 = \frac{1}{4}(2 + \sqrt{2})$, $s_{2\ell+1} \equiv s_1 = \frac{1}{4}(2 - \sqrt{2})$.

Assume that $\lambda_n = 1 + \epsilon$, where $\epsilon > 0$ is very small, so that the terms $\mathcal{O}(\epsilon^2)$ are negligible, and that $\mathbf{x}^{(0)}$ contains substantial components of all the eigenvectors.

By considering the approximation after $2m$ iterations in the form

$$\mathbf{x}^{(2m)} = \pm \mathbf{w}_n + \mathcal{O}(\rho^{2m}) \quad (m \to \infty),$$

find ρ as a function of ϵ for each of the three versions of the method.

Compare the convergence rates of the three versions of the method, with reference to the number of iterations needed to achieve a prescribed accuracy.

END OF PAPER