UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Tuesday, 6 June, 2017 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, …, ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

State and prove Legendre's formula for $\pi(x)$. Use it to compute $\pi(42)$.

2F Topics In Analysis

Are the following statements true or false? Give reasons, quoting any theorems that you need.

(i) There is a sequence of polynomials P_n with $P_n(t) \to \sin t$ uniformly on $\mathbb{R}$ as $n \to \infty$.

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then there is a sequence of polynomials Q_n with $Q_n(t) \to f(t)$ for each $t \in \mathbb{R}$ as $n \to \infty$.

(iii) If $g : [1, \infty) \to \mathbb{R}$ is continuous with $g(t) \to 0$ as $t \to \infty$, then there is a sequence of polynomials R_n with $R_n(1/t) \to g(t)$ uniformly on $[1, \infty)$ as $n \to \infty$.

3G Coding & Cryptography

Prove that a decipherable code with prescribed word lengths exists if and only if there is a prefix-free code with the same word lengths.

4H Automata and Formal Languages

(a) Give explicit examples, with justification, of a language over some finite alphabet Σ which is:

(i) context-free, but not regular;

(ii) recursive, but not context-free.

(b) Give explicit examples, with justification, of a subset of $\mathbb{N}$ which is:

(i) recursively enumerable, but not recursive;

(ii) neither recursively enumerable, nor having recursively enumerable complement in $\mathbb{N}$.

5J Statistical Modelling

A statistician is interested in the power of a t-test with level 5% in linear regression; that is, the probability of rejecting the null hypothesis $\beta_0 = 0$ with this test under an alternative with $\beta_0 > 0$.

(a) State the distribution of the least-squares estimator $\hat{\beta}_0$, and hence state the form of the t-test statistic used.

(b) Prove that the power does not depend on the other coefficients β_j for $j > 0$.

6B Mathematical Biology

A bacterial nutrient uptake model is represented by the reaction system

$$\begin{aligned} 2S + E &\xrightarrow{k_1} C \\ C &\xrightarrow{k_2} 2S + E \\ C &\xrightarrow{k_3} E + 2P \end{aligned}$$

where the k_i are rate constants. Let s, e, c and p represent the concentrations of S, E, C and P respectively. Initially $s = s_0$, $e = e_0$, $c = 0$ and $p = 0$. Write down the governing differential equation system for the concentrations.

Either by using the differential equations or directly from the reaction system above, find two invariant quantities. Use these to simplify the system to

$$\begin{aligned} \dot{s} &= -2k_1 s^2(e_0 - c) + 2k_2 c\,, \\ \dot{c} &= k_1 s^2(e_0 - c) - (k_2 + k_3)c\,. \end{aligned}$$

By setting $u = s/s_0$ and $v = c/e_0$ and rescaling time, show that the system can be written as

$$\begin{aligned} u' &= -2u^2(1 - v) + 2(\mu - \lambda)v\,, \\ \epsilon v' &= u^2(1 - v) - \mu v\,, \end{aligned}$$

where $\epsilon = e_0/s_0$ and μ and λ should be given. Give the initial conditions for u and v.

[*Hint: Note that 2X is equivalent to X+X in reaction systems.*]

7E Further Complex Methods

Euler's formula for the Gamma function is

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}.$$

Use Euler's formula to show

$$\frac{\Gamma(2z)}{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})} = C,$$

where C is a constant.

Evaluate C.

[*Hint: You may use* $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$.]

8E Classical Dynamics

Derive the Lagrange equations from the principle of stationary action

$$S[q] = \int_{t_0}^{t_1} \mathcal{L}(q_i(t), \dot{q}_i(t), t)dt, \quad \delta S = 0,$$

where the end points $q_i(t_0)$ and $q_i(t_1)$ are fixed.

Let ϕ and $\mathbf{A}$ be a scalar and a vector, respectively, depending on $\mathbf{r} = (x, y, z)$. Consider the Lagrangian

$$\mathcal{L} = \frac{m\dot{\mathbf{r}}^2}{2} - (\phi - \dot{\mathbf{r}} \cdot \mathbf{A}),$$

and show that the resulting Euler–Lagrange equations are invariant under the transformations

$$\phi \to \phi + \alpha\frac{\partial F}{\partial t}, \quad \mathbf{A} \to \mathbf{A} + \nabla F,$$

where $F = F(\mathbf{r}, t)$ is an arbitrary function, and α is a constant which should be determined.

9C Cosmology

In a homogeneous and isotropic universe ($\Lambda = 0$), the acceleration equation for the scale factor $a(t)$ is given by

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P/c^2\right),$$

where $\rho(t)$ is the mass density and $P(t)$ is the pressure.

If the matter content of the universe obeys the strong energy condition $\rho + 3P/c^2 \geqslant 0$, show that the acceleration equation can be rewritten as $\dot{H} + H^2 \leqslant 0$, with Hubble parameter $H(t) = \dot{a}/a$. Show that

$$H \geqslant \frac{1}{H_0^{-1} + t - t_0},$$

where $H_0 = H(t_0)$ is the measured value today at $t = t_0$. Hence, or otherwise, show that

$$a(t) \leqslant 1 + H_0(t - t_0).$$

Use this inequality to find an upper bound on the age of the universe.

SECTION II

10F Topics In Analysis

State and prove Baire's category theorem for complete metric spaces. Give an example to show that it may fail if the metric space is not complete.

Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions such that $f_n(x)$ converges for all $x \in [0,1]$. Show that if $\epsilon > 0$ is fixed we can find an $N \geqslant 0$ and a non-empty open interval $J \subseteq [0,1]$ such that $|f_n(x) - f_m(x)| \leqslant \epsilon$ for all $x \in J$ and all n, $m \geqslant N$.

Let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that we cannot find continuous functions $g_n : [0,1] \to \mathbb{R}$ with $g_n(x) \to g(x)$ for each $x \in [0,1]$ as $n \to \infty$.

Define a sequence of continuous functions $h_n : [0,1] \to \mathbb{R}$ and a discontinuous function $h : [0,1] \to \mathbb{R}$ with $h_n(x) \to h(x)$ for each $x \in [0,1]$ as $n \to \infty$.

11G Coding & Cryptography

Define the *entropy*, $H(X)$, of a random variable X. State and prove Gibbs' inequality.

Hence, or otherwise, show that $H(p_1, p_2, p_3) \leqslant H(p_1, 1-p_1) + (1-p_1)$ and determine when equality occurs.

Show that the Discrete Memoryless Channel with channel matrix

$$\begin{pmatrix} 1-\alpha-\beta & \alpha & \beta \\ \alpha & 1-\alpha-\beta & \beta \end{pmatrix}$$

has capacity $C = (1-\beta)(1-\log(1-\beta)) + (1-\alpha-\beta)\log(1-\alpha-\beta) + \alpha\log\alpha$.

12E Further Complex Methods

The hypergeometric equation is represented by the Papperitz symbol

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\} \qquad (*)$$

and has solution $y_0(z) = F(a,b,c;z)$.

Functions $y_1(z)$ and $y_2(z)$ are defined by

$$y_1(z) = F(a,b,a+b+1-c;1-z)$$

and

$$y_2(z) = (1-z)^{c-a-b} F(c-a,c-b,c-a-b+1;1-z),$$

where $c-a-b$ is not an integer.

Show that $y_1(z)$ and $y_2(z)$ obey the hypergeometric equation $(*)$.

Explain why $y_0(z)$ can be written in the form

$$y_0(z) = Ay_1(z) + By_2(z),$$

where A and B are independent of z but depend on a, b and c.

Suppose that

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$

with $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $|\arg(1-z)| < \pi$. Find expressions for A and B.

13E Classical Dynamics

Show that an object's inertia tensor about a point displaced from the centre of mass by a vector $\mathbf{c}$ is given by

$$(I_{\mathbf{c}})_{ab} = (I_0)_{ab} + M(|\mathbf{c}|^2\delta_{ab} - c_a c_b),$$

where M is the total mass of the object, and $(I_0)_{ab}$ is the inertia tensor about the centre of mass.

Find the inertia tensor of a cube of uniform density, with edge of length L, about one of its vertices.

14H Logic and Set Theory

Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent.

Which of the following are always true for ordinals α, β and γ and which can be false? Give proofs or counterexamples as appropriate.

(i) $\alpha + \beta = \beta + \alpha$

(ii) $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$

(iii) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

(iv) If $\alpha\beta = \beta\alpha$ then $\alpha^2\beta^2 = \beta^2\alpha^2$

(v) If $\alpha^2\beta^2 = \beta^2\alpha^2$ then $\alpha\beta = \beta\alpha$

[In parts (iv) and (v) you may assume without proof that ordinal multiplication is associative.]

15H Graph Theory

State and prove Hall's theorem about matchings in bipartite graphs.

Let $A = (a_{ij})$ be an $n \times n$ matrix, with all entries non-negative reals, such that every row sum and every column sum is 1. By applying Hall's theorem, show that there is a permutation σ of $\{1, \ldots, n\}$ such that $a_{i\sigma(i)} > 0$ for all i.

16I Galois Theory

(a) Define what it means for a finite field extension L of a field K to be *separable.* Show that L is of the form $K(\alpha)$ for some $\alpha \in L$.

(b) Let p and q be distinct prime numbers. Let $L = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$. Express L in the form $\mathbb{Q}(\alpha)$ and find the minimal polynomial of α over $\mathbb{Q}$.

(c) Give an example of a field extension $K \leqslant L$ of finite degree, where L is not of the form $K(\alpha)$. Justify your answer.

17G Representation Theory

In this question you may assume the following result. Let χ be a character of a finite group G and let $g \in G$. If $\chi(g)$ is a rational number, then $\chi(g)$ is an integer.

(a) If a and b are positive integers, we denote their highest common factor by (a, b). Let g be an element of order n in the finite group G. Suppose that g is conjugate to g^i for all i with $1 \leqslant i \leqslant n$ and $(i, n) = 1$. Prove that $\chi(g)$ is an integer for all characters χ of G.

[You may use the following result without proof. Let ω be an nth root of unity. Then

$$\sum_{\substack{1 \leqslant i \leqslant n, \\ (i,n)=1}} \omega^i$$

is an integer.]

Deduce that all the character values of symmetric groups are integers.

(b) Let G be a group of odd order.

Let χ be an irreducible character of G with $\chi = \bar{\chi}$. Prove that

$$\langle \chi, 1_G \rangle = \frac{1}{|G|}(\chi(1) + 2\alpha),$$

where α is an algebraic integer. Deduce that $\chi = 1_G$.

18H Number Fields

(a) Let L be a number field, $\mathcal{O}_L$ the ring of integers in L, $\mathcal{O}_L^*$ the units in $\mathcal{O}_L$, r the number of real embeddings of L, and s the number of pairs of complex embeddings of L.

Define a group homomorphism $\mathcal{O}_L^* \to \mathbb{R}^{r+s-1}$ with finite kernel, and prove that the image is a discrete subgroup of $\mathbb{R}^{r+s-1}$.

(b) Let $K = \mathbb{Q}(\sqrt{d})$ where $d > 1$ is a square-free integer. What is the structure of the group of units of K? Show that if d is divisible by a prime $p \equiv 3 \pmod 4$ then every unit of K has norm $+1$. Find an example of K with a unit of norm -1.

19I Algebraic Topology

(a) (i) Define the *push-out* of the following diagram of groups.

$$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"] & G_1 \\ G_2 & \end{tikzcd}$$

When is a push-out a *free product with amalgamation*?

(ii) State the Seifert–van Kampen theorem.

(b) Let $X = \mathbb{R}P^2 \vee S^1$ (recalling that $\mathbb{R}P^2$ is the real projective plane), and let $x \in X$.

(i) Compute the fundamental group $\pi_1(X, x)$ of the space X.

(ii) Show that there is a surjective homomorphism $\phi : \pi_1(X, x) \to S_3$, where S_3 is the symmetric group on three elements.

(c) Let $\widehat{X} \to X$ be the covering space corresponding to the kernel of ϕ.

(i) Draw $\widehat{X}$ and justify your answer carefully.

(ii) Does $\widehat{X}$ retract to a graph? Justify your answer briefly.

(iii) Does $\widehat{X}$ deformation retract to a graph? Justify your answer briefly.

20F Linear Analysis

(a) Let X be a normed vector space and $Y \subset X$ a closed subspace with $Y \neq X$. Show that Y is nowhere dense in X.

(b) State any version of the Baire Category theorem.

(c) Let X be an infinite-dimensional Banach space. Show that X cannot have a countable algebraic basis, i.e. there is no countable subset $(x_k)_{k \in \mathbb{N}} \subset X$ such that every $x \in X$ can be written as a finite linear combination of elements of (x_k).

21F Riemann Surfaces

Let f be a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$. Let P be a fundamental parallelogram whose boundary contains no zeros or poles of f. Show that the number of zeros of f in P is the same as the number of poles of f in P, both counted with multiplicities.

Suppose additionally that f is even. Show that there exists a rational function $Q(z)$ such that $f = Q(\wp)$, where $\wp$ is the Weierstrass $\wp$-function.

Suppose f is a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$, and F is a meromorphic antiderivative of f, so that $F' = f$. Is it necessarily true that F is an elliptic function? Justify your answer.

[You may use standard properties of the Weierstrass $\wp$-function throughout.]

22I Algebraic Geometry

Let k be an algebraically closed field of any characteristic.

(a) Define what it means for a variety X to be *non-singular* at a point $P \in X$.

(b) Let $X \subseteq \mathbb{P}^n$ be a hypersurface $Z(f)$ for $f \in k[x_0, \ldots, x_n]$ an irreducible homogeneous polynomial. Show that the set of singular points of X is $Z(I)$, where $I \subseteq k[x_0, \ldots, x_n]$ is the ideal generated by $\partial f/\partial x_0, \ldots, \partial f/\partial x_n$.

(c) Consider the projective plane curve corresponding to the affine curve in $\mathbb{A}^2$ given by the equation

$$x^4 + x^2y^2 + y^2 + 1 = 0.$$

Find the singular points of this projective curve if char $k \neq 2$. What goes wrong if char $k = 2$?

23I Differential Geometry

Let $\alpha : I \to \mathbb{R}^3$ be a regular smooth curve. Define the *curvature* k and *torsion* τ of α and derive the Frenet formulae. Give the assumption which must hold for torsion to be well-defined, and state the Fundamental Theorem for curves in $\mathbb{R}^3$.

Let α be as above and $\tilde{\alpha} : I \to \mathbb{R}^3$ be another regular smooth curve with curvature $\tilde{k}$ and torsion $\tilde{\tau}$. Suppose $\tilde{k}(s) = k(s) \neq 0$ and $\tilde{\tau}(s) = \tau(s)$ for all $s \in I$, and that there exists a non-empty open subinterval $J \subset I$ such that $\tilde{\alpha}|_J = \alpha|_J$. Show that $\tilde{\alpha} = \alpha$.

Now let $S \subset \mathbb{R}^3$ be an oriented surface and let $\alpha : I \to S \subset \mathbb{R}^3$ be a regular smooth curve contained in S. Define *normal curvature* and *geodesic curvature*. When is α a geodesic? Give an example of a surface S and a geodesic α whose normal curvature vanishes identically. Must such a surface S contain a piece of a plane? Can such a geodesic be a simple closed curve? Justify your answers.

Show that if α is a geodesic and the Gaussian curvature of S satisfies $K \geqslant 0$, then we have the inequality $k(s) \leqslant 2|H(\alpha(s))|$, where H denotes the mean curvature of S and k the curvature of α. Give an example where this inequality is sharp.

24J Probability and Measure

(a) Give the definition of the *Fourier transform* $\widehat{f}$ of a function $f \in L^1(\mathbb{R}^d)$.

(b) Explain what it means for Fourier inversion to hold.

(c) Prove that Fourier inversion holds for $g_t(x) = (2\pi t)^{-d/2} e^{-\|x\|^2/(2t)}$. Show all of the steps in your computation. Deduce that Fourier inversion holds for Gaussian convolutions, i.e. any function of the form $f * g_t$ where $t > 0$ and $f \in L^1(\mathbb{R}^d)$.

(d) Prove that any function f for which Fourier inversion holds has a bounded, continuous version. In other words, there exists g bounded and continuous such that $f(x) = g(x)$ for a.e. $x \in \mathbb{R}^d$.

(e) Does Fourier inversion hold for $f = \mathbf{1}_{[0,1]}$?

25K Applied Probability

(a) Give the definition of a *Poisson process* on $\mathbb{R}_+$. Let X be a Poisson process on $\mathbb{R}_+$. Show that conditional on $\{X_t = n\}$, the jump times $J_1, \ldots, J_n$ have joint density function

$$f(t_1, \ldots, t_n) = \frac{n!}{t^n} \mathbf{1}(0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant t)\,,$$

where $\mathbf{1}(A)$ is the indicator of the set A.

(b) Let N be a Poisson process on $\mathbb{R}_+$ with intensity λ and jump times $\{J_k\}$. If $g : \mathbb{R}_+ \to \mathbb{R}$ is a real function, we define for all $t > 0$

$$\mathcal{R}(g)[0,t] = \{g(J_k) : \ k \in \mathbb{N},\ J_k \leqslant t\}.$$

Show that for all $t > 0$ the following is true

$$\mathbb{P}(0 \in \mathcal{R}(g)[0,t]) = 1 - \exp\left(-\lambda \int_0^t \mathbf{1}(g(s) = 0)\, ds\right)\,.$$

26K Principles of Statistics

We consider the problem of estimating θ in the model $\{f(x,\theta) : \ \theta \in (0,\infty)\}$, where

$$f(x,\theta) = (1-\alpha)(x-\theta)^{-\alpha}\mathbf{1}\{x \in [\theta, \theta+1]\}\,.$$

Here $\mathbf{1}\{A\}$ is the indicator of the set A, and $\alpha \in (0,1)$ is known. This estimation is based on a sample of n i.i.d. $X_1, \ldots, X_n$, and we denote by $X_{(1)} < \ldots < X_{(n)}$ the ordered sample.

(a) Compute the mean and the variance of X_1. Construct an unbiased estimator of θ taking the form $\tilde{\theta}_n = \bar{X}_n + c(\alpha)$, where $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$, specifying $c(\alpha)$.

(b) Show that $\tilde{\theta}_n$ is consistent and find the limit in distribution of $\sqrt{n}(\tilde{\theta}_n - \theta)$. Justify your answer, citing theorems that you use.

(c) Find the maximum likelihood estimator $\hat{\theta}_n$ of θ. Compute $\mathbf{P}(\hat{\theta}_n - \theta > t)$ for all real t. Is $\hat{\theta}_n$ unbiased?

(d) For $t > 0$, show that $\mathbf{P}(n^\beta(\hat{\theta}_n - \theta) > t)$ has a limit in $(0,1)$ for some $\beta > 0$. Give explicitly the value of β and the limit. Why should one favour using $\hat{\theta}_n$ over $\tilde{\theta}_n$?

27J Stochastic Financial Models

(a) What is a *Brownian motion*?

(b) Let $(B_t, t \geqslant 0)$ be a Brownian motion. Show that the process $\tilde{B}_t := \frac{1}{c} B_{c^2 t}$, $c \in \mathbb{R} \setminus \{0\}$, is also a Brownian motion.

(c) Let $Z := \sup_{t \geqslant 0} B_t$. Show that $cZ \overset{(d)}{=} Z$ for all $c > 0$ (i.e. cZ and Z have the same laws). Conclude that $Z \in \{0, +\infty\}$ a.s.

(d) Show that $\mathbb{P}[Z = +\infty] = 1$.

28K Optimization and Control

During each of N time periods a venture capitalist, Vicky, is presented with an investment opportunity for which the rate of return for that period is a random variable; the rates of return in successive periods are independent identically distributed random variables with distributions concentrated on $[-1, \infty)$. Thus, if x_n is Vicky's capital at period n, then $x_{n+1} = (1 - p_n)x_n + p_n x_n(1 + R_n)$, where $p_n \in [0, 1]$ is the proportion of her capital she chooses to invest at period n, and R_n is the rate of return for period n. Vicky desires to maximize her expected yield over N periods, where the yield is defined as $\left(\frac{x_N}{x_0}\right)^{\frac{1}{N}} - 1$, and x_0 and x_N are respectively her initial and final capital.

(a) Express the problem of finding an optimal policy in a dynamic programming framework.

(b) Show that in each time period, the optimal strategy can be expressed in terms of the quantity p^* which solves the optimization problem $\max_p \mathbb{E}(1 + pR_1)^{1/N}$. Show that $p^* > 0$ if $\mathbb{E}R_1 > 0$. [Do not calculate p^* explicitly.]

(c) Compare her optimal policy with the policy which maximizes her expected final capital x_N.

29E Asymptotic Methods

Consider the function

$$f_\nu(x) \equiv \frac{1}{2\pi} \int_C \exp\left[-ix \sin z + i\nu z\right] dz,$$

where the contour C is the boundary of the half-strip $\{z : -\pi < \text{Re } z < \pi \text{ and Im } z > 0\}$, taken anti-clockwise.

Use integration by parts and the method of stationary phase to:

(i) Obtain the leading term for $f_\nu(x)$ coming from the vertical lines $z = \pm\pi + iy$ $(0 < y < +\infty)$ for large $x > 0$.

(ii) Show that the leading term in the asymptotic expansion of the function $f_\nu(x)$ for large positive x is

$$\sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right),$$

and obtain an estimate for the remainder as $O(x^{-a})$ for some a to be determined.

30A Dynamical Systems

(a) State Liapunov's first theorem and La Salle's invariance principle. Use these results to show that the fixed point at the origin of the system

$$\ddot{x} + k\dot{x} + \sin^3 x = 0\,, \quad k > 0,$$

is asymptotically stable.

(b) State the Poincaré–Bendixson theorem. Show that the forced damped pendulum

$$\dot{\theta} = p, \qquad \dot{p} = -kp - \sin\theta + F\,, \quad k > 0, \qquad (*)$$

with $F > 1$, has a periodic orbit that encircles the cylindrical phase space $(\theta, p) \in \mathbb{R}[\text{mod } 2\pi] \times \mathbb{R}$, and that it is unique.

[You may assume that the Poincaré–Bendixson theorem also holds on a cylinder, and comment, without proof, on the use of any other standard results.]

(c) Now consider $(*)$ for $F, k = O(\epsilon)$, where $\epsilon \ll 1$. Use the energy-balance method to show that there is a homoclinic orbit in $p \geqslant 0$ if $F = F_h(k)$, where $F_h \approx 4k/\pi > 0$.

Explain briefly why there is no homoclinic orbit in $p \leqslant 0$ for $F > 0$.

31A Integrable Systems

Let U and V be non-singular $N \times N$ matrices depending on (x, t, λ) which are periodic in x with period 2π. Consider the associated linear problem

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi\,,$$

for the vector $\Psi = \Psi(x, t; \lambda)$. On the assumption that these equations are compatible, derive the zero curvature equation for (U, V).

Let $W = W(x, t, \lambda)$ denote the $N \times N$ matrix satisfying

$$W_x = UW, \quad W(0, t, \lambda) = I_N,$$

where I_N is the $N \times N$ identity matrix. You should assume W is unique. By considering $(W_t - VW)_x$, show that the matrix $w(t, \lambda) = W(2\pi, t, \lambda)$ satisfies the Lax equation

$$w_t = [v, w], \quad v(t, \lambda) \equiv V(2\pi, t, \lambda).$$

Deduce that $\left\{\operatorname{tr}\left(w^k\right)\right\}_{k \geqslant 1}$ are first integrals.

By considering the matrices

$$\frac{1}{2\mathrm{i}\lambda}\begin{bmatrix}\cos u & -\mathrm{i}\sin u\\ \mathrm{i}\sin u & -\cos u\end{bmatrix}, \quad \frac{\mathrm{i}}{2}\begin{bmatrix}2\lambda & u_x\\ u_x & -2\lambda\end{bmatrix},$$

show that the periodic Sine-Gordon equation $u_{xt} = \sin u$ has infinitely many first integrals. [You need not prove anything about independence.]

32C Principles of Quantum Mechanics

Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be a set of Hermitian operators obeying

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \text{and} \quad (\mathbf{n}\cdot\boldsymbol{\sigma})^2 = 1, \tag{$*$}$$

where $\mathbf{n}$ is any unit vector. Show that $(*)$ implies that

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = \mathbf{a}\cdot\mathbf{b} + i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma},$$

for any vectors $\mathbf{a}$ and $\mathbf{b}$. Explain, with reference to the properties $(*)$, how $\boldsymbol{\sigma}$ can be related to the intrinsic angular momentum $\mathbf{S}$ for a particle of spin $\frac{1}{2}$.

Show that the operators $P_\pm = \frac{1}{2}(1 \pm \mathbf{n}\cdot\boldsymbol{\sigma})$ are Hermitian and obey

$$P_\pm^2 = P_\pm, \quad P_+P_- = P_-P_+ = 0.$$

Show how $P_\pm$ can be used to write any state $|\chi\rangle$ as a linear combination of eigenstates of $\mathbf{n}\cdot\boldsymbol{\sigma}$. Use this to deduce that if the system is in a normalised state $|\chi\rangle$ when $\mathbf{n}\cdot\boldsymbol{\sigma}$ is measured, then the results ± 1 will be obtained with probabilities

$$\|P_\pm|\chi\rangle\|^2 = \frac{1}{2}(1 \pm \langle\chi|\mathbf{n}\cdot\boldsymbol{\sigma}|\chi\rangle).$$

If $|\chi\rangle$ is a state corresponding to the system having spin up along a direction defined by a unit vector $\mathbf{m}$, show that a measurement will find the system to have spin up along $\mathbf{n}$ with probability $\frac{1}{2}(1 + \mathbf{n}\cdot\mathbf{m})$.

33C Applications of Quantum Mechanics

Give an account of the variational method for establishing an upper bound on the ground-state energy of a Hamiltonian H with a discrete spectrum $H|n\rangle = E_n|n\rangle$, where $E_n \leqslant E_{n+1}$, $n = 0, 1, 2 \ldots$.

A particle of mass m moves in the three-dimensional potential

$$V(r) = -\frac{Ae^{-\mu r}}{r},$$

where $A, \mu > 0$ are constants and r is the distance to the origin. Using the normalised variational wavefunction

$$\psi(r) = \sqrt{\frac{\alpha^3}{\pi}} e^{-\alpha r},$$

show that the expected energy is given by

$$E(\alpha) = \frac{\hbar^2 \alpha^2}{2m} - \frac{4A\alpha^3}{(\mu + 2\alpha)^2}.$$

Explain why there is necessarily a bound state when $\mu < Am/\hbar^2$. What can you say about the existence of a bound state when $\mu \geqslant Am/\hbar^2$?

[*Hint: The Laplacian in spherical polar coordinates is*

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}. \quad]$$

34D Statistical Physics

(a) The entropy of a thermodynamic ensemble is defined by the formula

$$S = -k \sum_n p(n) \log p(n)\,,$$

where k is the Boltzmann constant. Explain what is meant by $p(n)$ in this formula. Write down an expression for $p(n)$ in the grand canonical ensemble, defining any variables you need. Hence show that the entropy S is related to the grand canonical partition function $\mathcal{Z}(T, \mu, V)$ by

$$S = k \left[\frac{\partial}{\partial T} (T \log \mathcal{Z}) \right]_{\mu, V}\,.$$

(b) Consider a gas of non-interacting fermions with single-particle energy levels ϵ_i.

(i) Show that the grand canonical partition function $\mathcal{Z}$ is given by

$$\log \mathcal{Z} = \sum_i \log \left(1 + e^{-(\epsilon_i - \mu)/(kT)} \right)\,.$$

(ii) Assume that the energy levels are continuous with density of states $g(\epsilon) = AV\epsilon^a$, where A and a are positive constants. Prove that

$$\log \mathcal{Z} = VT^b f(\mu/T)$$

and give expressions for the constant b and the function f.

(iii) The gas is isolated and undergoes a reversible adiabatic change. By considering the ratio S/N, prove that μ/T remains constant. Deduce that VT^c and pV^d remain constant in this process, where c and d are constants whose values you should determine.

35D General Relativity

(a) The Friedmann–Robertson–Walker metric is given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right],$$

where $k = -1, 0, +1$ and $a(t)$ is the scale factor.

For $k = +1$, show that this metric can be written in the form

$$ds^2 = -dt^2 + \gamma_{ij}dx^i dx^j = -dt^2 + a^2(t)\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\right].$$

Calculate the equatorial circumference ($\theta = \pi/2$) of the submanifold defined by constant t and χ.

Calculate the proper volume, defined by $\int \sqrt{\det\gamma}\; d^3x$, of the hypersurface defined by constant t.

(b) The Friedmann equations are

$$3\left(\frac{\dot{a}^2 + k}{a^2}\right) - \Lambda = 8\pi\rho\,,$$
$$\frac{2a\ddot{a} + \dot{a}^2 + k}{a^2} - \Lambda = -8\pi P\,,$$

where $\rho(t)$ is the energy density, $P(t)$ is the pressure, Λ is the cosmological constant and dot denotes d/dt.

The Einstein static universe has vanishing pressure, $P(t) = 0$. Determine a, k and Λ as a function of the density ρ.

The Einstein static universe with $a = a_0$ and $\rho = \rho_0$ is perturbed by radiation such that

$$a = a_0 + \delta a(t)\,, \qquad \rho = \rho_0 + \delta\rho(t)\,, \qquad P = \frac{1}{3}\delta\rho(t)\,,$$

where $\delta a \ll a_0$ and $\delta\rho \ll \rho_0$. Show that the Einstein static universe is unstable to this perturbation.

36B Fluid Dynamics II

A cylinder of radius a falls at speed U without rotating through viscous fluid adjacent to a vertical plane wall, with its axis horizontal and parallel to the wall. The distance between the cylinder and the wall is $h_0 \ll a$. Use lubrication theory in a frame of reference moving with the cylinder to determine that the two-dimensional volume flux between the cylinder and the wall is

$$q = \frac{2h_0 U}{3}$$

upwards, relative to the cylinder.

Determine an expression for the viscous shear stress on the cylinder. Use this to calculate the viscous force and hence the torque on the cylinder. If the cylinder is free to rotate, what does your result say about the sense of rotation of the cylinder?

[*Hint: You may quote the following integrals:*

$$\int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \pi, \qquad \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^2} = \frac{\pi}{2}, \qquad \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^3} = \frac{3\pi}{8}. \quad]$$

37B Waves

Show that, for a one-dimensional flow of a perfect gas (with $\gamma > 1$) at constant entropy, the Riemann invariants $R_{\pm} = u \pm 2(c - c_0)/(\gamma - 1)$ are constant along characteristics $dx/dt = u \pm c$.

Define a *simple wave*. Show that in a right-propagating simple wave

$$\frac{\partial u}{\partial t} + \left(c_0 + \tfrac{1}{2}(\gamma+1)u\right)\frac{\partial u}{\partial x} = 0 \,.$$

In some circumstances, dissipative effects may be modelled by

$$\frac{\partial u}{\partial t} + \left(c_0 + \tfrac{1}{2}(\gamma+1)u\right)\frac{\partial u}{\partial x} = -\alpha u \,,$$

where α is a positive constant. Suppose also that u is prescribed at $t = 0$ for all x, say $u(x,0) = u_0(x)$. Demonstrate that, unless a shock develops, a solution of the form

$$u(x,t) = u_0(\xi)e^{-\alpha t}$$

can be found, where, for each x and t, ξ is determined implicitly as the solution of the equation

$$x - c_0 t = \xi + \frac{\gamma+1}{2\alpha}\left(1 - e^{-\alpha t}\right)u_0(\xi) \,.$$

Deduce that, despite the presence of dissipative effects, a shock will still form at some (x,t) unless $\alpha > \alpha_c$, where

$$\alpha_c = \tfrac{1}{2}(\gamma+1)\max_{u_0'<0}|u_0'(\xi)| \,.$$

38A Numerical Analysis

The Poisson equation $\nabla^2 u = f$ in the unit square $\Omega = [0,1] \times [0,1]$, equipped with the zero Dirichlet boundary conditions on $\partial\Omega$, is discretized with the nine-point formula:

$$\begin{aligned}\Gamma_9[u_{i,j}] := -\frac{10}{3}\, u_{i,j} + \frac{2}{3}\,(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) \\ +\frac{1}{6}\,(u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1}) = h^2 f_{i,j},\end{aligned}$$

where $1 \leqslant i,j \leqslant m$, $u_{i,j} \approx u(ih, jh)$, and (ih, jh) are the grid points with $h = \frac{1}{m+1}$.

(i) Find the order of the local truncation error $\eta_{i,j}$ of the approximation.

(ii) Prove that the order of the truncation error is smaller if f satisfies the Laplace equation $\nabla^2 f = 0$.

(iii) Show that the modified nine-point scheme

$$\begin{aligned}\Gamma_9[u_{i,j}] &= h^2 f_{i,j} + \frac{1}{12} h^2 \Gamma_5[f_{i,j}] \\ &:= h^2 f_{i,j} + \frac{1}{12} h^2 (f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4 f_{i,j})\end{aligned}$$

has a truncation error of the same order as in part (ii).

(iv) Let $(u_{i,j})_{i,j=1}^{m}$ be a solution to the $m^2 \times m^2$ system of linear equations $A\mathbf{u} = \mathbf{b}$ arising from the modified nine-point scheme in part (iii). Further, let $u(x,y)$ be the exact solution and let $e_{i,j} := u_{i,j} - u(ih, jh)$ be the error of approximation at grid points. Prove that there exists a constant c such that

$$\|\mathbf{e}\|_2 := \left[\sum_{i,j=1}^{m} |e_{i,j}|^2\right]^{1/2} < ch^3, \qquad h \to 0.$$

[*Hint: The nine-point discretization of* $\nabla^2 u$ *can be written as*

$$\Gamma_9[u] = (\Gamma_5 + \tfrac{1}{6}\Delta_x^2 \Delta_y^2)\, u = (\Delta_x^2 + \Delta_y^2 + \tfrac{1}{6}\Delta_x^2\Delta_y^2)\, u\,,$$

where $\Gamma_5[u] = (\Delta_x^2 + \Delta_y^2) u$ *is the five-point discretization and*

$$\begin{aligned}\Delta_x^2\, u(x,y) &:= u(x-h,y) - 2u(x,y) + u(x+h,y), \\ \Delta_y^2\, u(x,y) &:= u(x,y-h) - 2u(x,y) + u(x,y+h)\,.\end{aligned}$$]

[*Hint: The matrix* A *of the nine-point scheme is symmetric, with the eigenvalues*

$$\lambda_{k,\ell} = -\,4\sin^2\tfrac{k\pi h}{2} - 4\sin^2\tfrac{\ell\pi h}{2} + \tfrac{8}{3}\sin^2\tfrac{k\pi h}{2}\sin^2\tfrac{\ell\pi h}{2}\,, \qquad 1 \leqslant k,\ell \leqslant m\,.$$]

END OF PAPER