UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2017

List of Courses

Paper 1, Section I
3F Analysis I

Given an increasing sequence of non-negative real numbers $(a_n)_{n=1}^{\infty}$, let

$$s_n = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Prove that if $s_n \to x$ as $n \to \infty$ for some $x \in \mathbb{R}$ then also $a_n \to x$ as $n \to \infty$.

Paper 1, Section II
11F Analysis I

(a) Let $(x_n)_{n=1}^{\infty}$ be a non-negative and decreasing sequence of real numbers. Prove that $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k x_{2^k}$ converges.

(b) For $s \in \mathbb{R}$, prove that $\sum_{n=1}^{\infty} n^{-s}$ converges if and only if $s > 1$.

(c) For any $k \in \mathbb{N}$, prove that

$$\lim_{n\to\infty} 2^{-n} n^k = 0.$$

(d) The sequence $(a_n)_{n=0}^{\infty}$ is defined by $a_0 = 1$ and $a_{n+1} = 2^{a_n}$ for $n \geqslant 0$. For any $k \in \mathbb{N}$, prove that

$$\lim_{n\to\infty} \frac{2^{n^k}}{a_n} = 0.$$

Paper 1, Section I

4E Analysis I

Show that if the power series $\sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{C})$ converges for some fixed $z = z_0$, then it converges absolutely for every z satisfying $|z| < |z_0|$.

Define the *radius of convergence* of a power series.

Give an example of $v \in \mathbb{C}$ and an example of $w \in \mathbb{C}$ such that $|v| = |w| = 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{v^n}{n}$ converges and $\displaystyle\sum_{n=1}^{\infty} \frac{w^n}{n}$ diverges. [You may assume results about standard series without proof.] Use this to find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$.

Paper 1, Section II

9D Analysis I

(a) State the Intermediate Value Theorem.

(b) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *differentiable* at a point $a \in \mathbb{R}$. If f is differentiable everywhere on $\mathbb{R}$, must f' be continuous everywhere? Justify your answer.

State the Mean Value Theorem.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Let $a, b \in \mathbb{R}$ with $a < b$.
If $f'(a) \leqslant y \leqslant f'(b)$, prove that there exists $c \in [a, b]$ such that $f'(c) = y$. [*Hint: consider the function g defined by*

$$g(x) = \frac{f(x) - f(a)}{x - a}$$

if $x \neq a$ and $g(a) = f'(a)$.]

If additionally $f(a) \leqslant 0 \leqslant f(b)$, deduce that there exists $d \in [a, b]$ such that $f'(d) + f(d) = y$.

Paper 1, Section II

10D Analysis I

Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : (a, b) \to \mathbb{R}$.

(a) Define what it means for f to be *continuous* at $y_0 \in (a, b)$.

f is said to have a *local minimum* at $c \in (a, b)$ if there is some $\varepsilon > 0$ such that $f(c) \leqslant f(x)$ whenever $x \in (a, b)$ and $|x - c| < \varepsilon$.

If f has a local minimum at $c \in (a, b)$ and f is differentiable at c, show that $f'(c) = 0$.

(b) f is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y)$$

for every $x, y \in (a, b)$ and $\lambda \in [0, 1]$. If f is convex, $r \in \mathbb{R}$ and $\big[y_0 - |r|\, ,\, y_0 + |r|\big] \subset (a, b)$, prove that

$$(1 + \lambda)f(y_0) - \lambda f(y_0 - r) \leqslant f(y_0 + \lambda r) \leqslant (1 - \lambda)f(y_0) + \lambda f(y_0 + r)$$

for every $\lambda \in [0, 1]$.

Deduce that if f is convex then f is continuous.

If f is convex and has a local minimum at $c \in (a, b)$, prove that f has a global minimum at c, i.e., that $f(x) \geqslant f(c)$ for every $x \in (a, b)$. [*Hint: argue by contradiction.*] Must f be differentiable at c? Justify your answer.

Paper 1, Section II

12E Analysis I

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function defined on the closed, bounded interval $[a,b]$ of $\mathbb{R}$. Suppose that for every $\varepsilon > 0$ there is a dissection $\mathcal{D}$ of $[a,b]$ such that $S_{\mathcal{D}}(f) - s_{\mathcal{D}}(f) < \varepsilon$, where $s_{\mathcal{D}}(f)$ and $S_{\mathcal{D}}(f)$ denote the lower and upper Riemann sums of f for the dissection $\mathcal{D}$. Deduce that f is Riemann integrable. [You may assume without proof that $s_{\mathcal{D}}(f) \leqslant S_{\mathcal{D}'}(f)$ for all dissections $\mathcal{D}$ and $\mathcal{D}'$ of $[a,b]$.]

Prove that if $f\colon [a,b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.

Let $g\colon (0,1] \to \mathbb{R}$ be a bounded continuous function. Show that for any $\lambda \in \mathbb{R}$, the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} g(x) & \text{if } 0 < x \leqslant 1 \ , \\ \lambda & \text{if } x = 0 \ , \end{cases}$$

is Riemann integrable.

Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function with one-sided derivatives at the endpoints. Suppose that the derivative f' is (bounded and) Riemann integrable. Show that

$$\int_a^b f'(x)\,dx = f(b) - f(a) \ .$$

[*You may use the Mean Value Theorem without proof.*]

Paper 2, Section I
1C Differential Equations

(a) The numbers $z_1, z_2, \ldots$ satisfy

$$z_{n+1} = z_n + c_n \qquad (n \geqslant 1),$$

where $c_1, c_2, \ldots$ are given constants. Find z_{n+1} in terms of $c_1, c_2, \ldots, c_n$ and z_1.

(b) The numbers $x_1, x_2, \ldots$ satisfy

$$x_{n+1} = a_n x_n + b_n \qquad (n \geqslant 1),$$

where $a_1, a_2, \ldots$ are given non-zero constants and $b_1, b_2, \ldots$ are given constants. Let $z_1 = x_1$ and $z_{n+1} = x_{n+1}/U_n$, where $U_n = a_1 a_2 \cdots a_n$. Calculate $z_{n+1} - z_n$, and hence find x_{n+1} in terms of x_1, $b_1, \ldots, b_n$ and $U_1, \ldots, U_n$.

Paper 2, Section I
2C Differential Equations

Consider the function

$$f(x,y) = \frac{x}{y} + \frac{y}{x} - \frac{(x-y)^2}{a^2}$$

defined for $x > 0$ and $y > 0$, where a is a non-zero real constant. Show that (λ, λ) is a stationary point of f for each $\lambda > 0$. Compute the Hessian and its eigenvalues at (λ, λ).

Paper 2, Section II

5C Differential Equations

The current $I(t)$ at time t in an electrical circuit subject to an applied voltage $V(t)$ obeys the equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}\,,$$

where R, L and C are the constant resistance, inductance and capacitance of the circuit with $R \geqslant 0$, $L > 0$ and $C > 0$.

(a) In the case $R = 0$ and $V(t) = 0$, show that there exist time-periodic solutions of frequency ω_0, which you should find.

(b) In the case $V(t) = H(t)$, the Heaviside function, calculate, subject to the condition

$$R^2 > \frac{4L}{C}\,,$$

the current for $t \geqslant 0$, assuming it is zero for $t < 0$.

(c) If $R > 0$ and $V(t) = \sin\omega_0 t$, where ω_0 is as in part (a), show that there is a time-periodic solution $I_0(t)$ of period $T = 2\pi/\omega_0$ and calculate its maximum value I_M.

(i) Calculate the energy dissipated in each period, i.e., the quantity

$$D = \int_0^T RI_0(t)^2\,dt\,.$$

Show that the quantity defined by

$$Q = \frac{2\pi}{D} \times \frac{LI_M^2}{2}$$

satisfies $Q\omega_0 RC = 1$.

(ii) Write down explicitly the general solution $I(t)$ for all $R > 0$, and discuss the relevance of $I_0(t)$ to the large time behaviour of $I(t)$.

Paper 2, Section II
6C Differential Equations

(a) Consider the system

$$\frac{dx}{dt} = x(1-x) - xy$$

$$\frac{dy}{dt} = \frac{1}{8}\,y(4x-1)$$

for $x(t) \geqslant 0$, $y(t) \geqslant 0$. Find the critical points, determine their type and explain, with the help of a diagram, the behaviour of solutions for large positive times t.

(b) Consider the system

$$\frac{dx}{dt} = y + (1-x^2-y^2)x$$

$$\frac{dy}{dt} = -x + (1-x^2-y^2)y$$

for $(x(t), y(t)) \in \mathbb{R}^2$. Rewrite the system in polar coordinates by setting $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$, and hence describe the behaviour of solutions for large positive and large negative times.

Paper 2, Section II

7C Differential Equations

Let y_1 and y_2 be two solutions of the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0\,, \qquad -\infty < x < \infty,$$

where p and q are given. Show, using the Wronskian, that

- *either* there exist α and β, not both zero, such that $\alpha y_1(x)+\beta y_2(x)$ vanishes for all x,
- *or* given x_0, A and B, there exist a and b such that $y(x) = ay_1(x) + by_2(x)$ satisfies the conditions $y(x_0) = A$ and $y'(x_0) = B$.

Find power series y_1 and y_2 such that an arbitrary solution of the equation

$$y''(x) = xy(x)$$

can be written as a linear combination of y_1 and y_2.

Paper 2, Section II
8C Differential Equations

(a) Solve $\dfrac{dz}{dt} = z^2$ subject to $z(0) = z_0$. For which z_0 is the solution finite for all $t \in \mathbb{R}$?

Let a be a positive constant. By considering the lines $y = a(x - x_0)$ for constant x_0, or otherwise, show that any solution of the equation

$$\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial y} = 0$$

is of the form $f(x, y) = F(y - ax)$ for some function F.

Solve the equation

$$\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial y} = f^2$$

subject to $f(0, y) = g(y)$ for a given function g. For which g is the solution bounded on $\mathbb{R}^2$?

(b) By means of the change of variables $X = \alpha x + \beta y$ and $T = \gamma x + \delta y$ for appropriate real numbers $\alpha, \beta, \gamma, \delta$, show that the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} = 0 \qquad (*)$$

can be transformed into the wave equation

$$\frac{1}{c^2}\frac{\partial^2 F}{\partial T^2} - \frac{\partial^2 F}{\partial X^2} = 0,$$

where F is defined by $f(x, y) = F(\alpha x + \beta y, \gamma x + \delta y)$. Hence write down the general solution of $(*)$.

Paper 4, Section I

3A Dynamics and Relativity

Consider a system of particles with masses m_i and position vectors $\mathbf{x}_i$. Write down the definition of the position of the *centre of mass* $\mathbf{R}$ of the system. Let T be the total kinetic energy of the system. Show that

$$T = \frac{1}{2}M\dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \frac{1}{2}\sum_i m_i \dot{\mathbf{y}}_i \cdot \dot{\mathbf{y}}_i \,,$$

where M is the total mass and $\mathbf{y}_i$ is the position vector of particle i with respect to $\mathbf{R}$.

The particles are connected to form a rigid body which rotates with angular speed ω about an axis $\mathbf{n}$ through $\mathbf{R}$, where $\mathbf{n} \cdot \mathbf{n} = 1$. Show that

$$T = \frac{1}{2}M\dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \frac{1}{2}I\omega^2 \,,$$

where $I = \sum_i I_i$ and I_i is the moment of inertia of particle i about $\mathbf{n}$.

Paper 4, Section I

4A Dynamics and Relativity

A tennis ball of mass m is projected vertically upwards with initial speed u_0 and reaches its highest point at time T. In addition to uniform gravity, the ball experiences air resistance, which produces a frictional force of magnitude αv, where v is the ball's speed and α is a positive constant. Show by dimensional analysis that T can be written in the form

$$T = \frac{m}{\alpha}\, f(\lambda)$$

for some function f of a dimensionless quantity λ.

Use the equation of motion of the ball to find $f(\lambda)$.

Paper 4, Section II

9A Dynamics and Relativity

(a) A photon with energy E_1 in the laboratory frame collides with an electron of rest mass m that is initially at rest in the laboratory frame. As a result of the collision the photon is deflected through an angle θ as measured in the laboratory frame and its energy changes to E_2.

Derive an expression for $\dfrac{1}{E_2} - \dfrac{1}{E_1}$ in terms of θ, m and c.

(b) A deuterium atom with rest mass m_1 and energy E_1 in the laboratory frame collides with another deuterium atom that is initially at rest in the laboratory frame. The result of this collision is a proton of rest mass m_2 and energy E_2, and a tritium atom of rest mass m_3. Show that, if the proton is emitted perpendicular to the incoming trajectory of the deuterium atom as measured in the laboratory frame, then

$$m_3^2 = m_2^2 + 2\left(m_1 + \frac{E_1}{c^2}\right)\left(m_1 - \frac{E_2}{c^2}\right).$$

Paper 4, Section II

10A Dynamics and Relativity

A particle of unit mass moves under the influence of a central force. By considering the components of the acceleration in polar coordinates (r, θ) prove that the magnitude of the angular momentum is conserved. [You may use $\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}$.]

Now suppose that the central force is derived from the potential k/r, where k is a constant.

(a) Show that the total energy of the particle can be written in the form

$$E = \tfrac{1}{2}\dot{r}^2 + V_{\text{eff}}(r).$$

Sketch $V_{\text{eff}}(r)$ in the cases $k > 0$ and $k < 0$.

(b) The particle is projected from a very large distance from the origin with speed v and impact parameter b. [The *impact parameter* is the distance of closest approach to the origin in absence of any force.]

(i) In the case $k < 0$, sketch the particle's trajectory and find the shortest distance p between the particle and the origin, and the speed u of the particle when $r = p$.

(ii) In the case $k > 0$, sketch the particle's trajectory and find the corresponding shortest distance $\widetilde{p}$ between the particle and the origin, and the speed $\widetilde{u}$ of the particle when $r = \widetilde{p}$.

(iii) Find $p\widetilde{p}$ and $u\widetilde{u}$ in terms of b and v. [In answering part (iii) you should assume that $|k|$ is the same in parts (i) and (ii).]

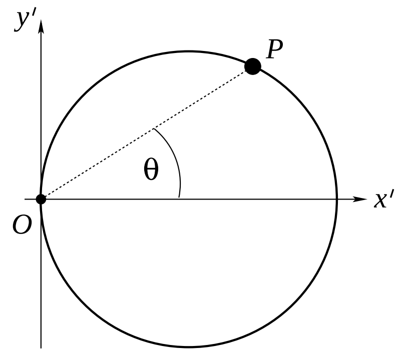
Paper 4, Section II
11A Dynamics and Relativity

(a) Consider an inertial frame S, and a frame S' which rotates with constant angular velocity $\boldsymbol{\omega}$ relative to S. The two frames share a common origin. Identify each term in the equation

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S'} = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_S - 2\,\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{S'} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

(b) A small bead P of unit mass can slide without friction on a circular hoop of radius a. The hoop is horizontal and rotating with constant angular speed ω about a fixed vertical axis through a point O on its circumference.

(i) Using Cartesian axes in the rotating frame S', with origin at O and x'-axis along the diameter of the hoop through O, write down the position vector of P in terms of a and the angle θ shown in the diagram $(-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi)$.



(ii) Working again in the rotating frame, find, in terms of a, θ, $\dot{\theta}$ and ω, an expression for the horizontal component of the force exerted by the hoop on the bead.

(iii) For what value of θ is the bead in stable equilibrium? Find the frequency of small oscillations of the bead about that point.

Paper 4, Section II
12A Dynamics and Relativity

(a) A rocket moves in a straight line with speed $v(t)$ and is subject to no external forces. The rocket is composed of a body of mass M and fuel of mass $m(t)$, which is burnt at constant rate α and the exhaust is ejected with constant speed u relative to the rocket. Show that

$$(M+m)\frac{dv}{dt} - \alpha u = 0\,.$$

Show that the speed of the rocket when all its fuel is burnt is

$$v_0 + u\log\left(1+\frac{m_0}{M}\right)\,,$$

where v_0 and m_0 are the speed of the rocket and the mass of the fuel at $t=0$.

(b) A two-stage rocket moves in a straight line and is subject to no external forces. The rocket is initially at rest. The masses of the bodies of the two stages are kM and $(1-k)M$, with $0 \leqslant k \leqslant 1$, and they initially carry masses km_0 and $(1-k)m_0$ of fuel. Both stages burn fuel at a constant rate α when operating and the exhaust is ejected with constant speed u relative to the rocket. The first stage operates first, until all its fuel is burnt. The body of the first stage is then detached with negligible force and the second stage ignites.

Find the speed of the second stage when all its fuel is burnt. For $0 \leqslant k < 1$ compare it with the speed of the rocket in part (a) in the case $v_0 = 0$. Comment on the case $k=1$.

Paper 3, Section I

1E Groups

Let w_1, w_2, w_3 be distinct elements of $\mathbb{C} \cup \{\infty\}$. Write down the Möbius map f that sends w_1, w_2, w_3 to $\infty, 0, 1$, respectively. [*Hint: You need to consider four cases.*]

Now let w_4 be another element of $\mathbb{C} \cup \{\infty\}$ distinct from w_1, w_2, w_3. Define the *cross-ratio* $[w_1, w_2, w_3, w_4]$ in terms of f.

Prove that there is a circle or line through w_1, w_2, w_3 and w_4 if and only if the cross-ratio $[w_1, w_2, w_3, w_4]$ is real.

[*You may assume without proof that Möbius maps map circles and lines to circles and lines and also that there is a unique circle or line through any three distinct points of* $\mathbb{C} \cup \{\infty\}$.]

Paper 3, Section I

2E Groups

What does it mean to say that H is a *normal subgroup* of the group G? For a normal subgroup H of G define the quotient group G/H. [You do not need to verify that G/H is a group.]

State the Isomorphism Theorem.

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R},\ ad \neq 0 \right\}$$

be the group of 2×2 invertible upper-triangular real matrices. By considering a suitable homomorphism, show that the subset

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\}$$

of G is a normal subgroup of G and identify the quotient G/H.

Paper 3, Section II

5E Groups

Let N be a normal subgroup of a finite group G of prime index $p = |G : N|$.

By considering a suitable homomorphism, show that if H is a subgroup of G that is not contained in N, then $H \cap N$ is a normal subgroup of H of index p.

Let C be a conjugacy class of G that is contained in N. Prove that C is either a conjugacy class in N or is the disjoint union of p conjugacy classes in N.

[*You may use standard theorems without proof.*]

Paper 3, Section II

6E Groups

State Lagrange's theorem. Show that the order of an element x in a finite group G is finite and divides the order of G.

State Cauchy's theorem.

List all groups of order 8 up to isomorphism. Carefully justify that the groups on your list are pairwise non-isomorphic and that any group of order 8 is isomorphic to one on your list. [You may use without proof the Direct Product Theorem and the description of standard groups in terms of generators satisfying certain relations.]

Paper 3, Section II
7E Groups

(a) Let G be a finite group acting on a finite set X. State the Orbit-Stabiliser theorem. [Define the terms used.] Prove that

$$\sum_{x \in X} |\mathrm{Stab}(x)| = n|G| \ ,$$

where n is the number of distinct orbits of X under the action of G.

Let $S = \{(g, x) \in G \times X : g \cdot x = x\}$, and for $g \in G$, let $\mathrm{Fix}(g) = \{x \in X : g \cdot x = x\}$. Show that

$$|S| = \sum_{x \in X} |\mathrm{Stab}(x)| = \sum_{g \in G} |\mathrm{Fix}(g)| \ ,$$

and deduce that

$$n = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)| \ . \qquad (*)$$

(b) Let H be the group of rotational symmetries of the cube. Show that H has 24 elements. [If your proof involves calculating stabilisers, then you must carefully verify such calculations.]

Using $(*)$, find the number of distinct ways of colouring the faces of the cube red, green and blue, where two colourings are distinct if one cannot be obtained from the other by a rotation of the cube. [A colouring need not use all three colours.]

Paper 3, Section II
8E Groups

Prove that every element of the symmetric group S_n is a product of transpositions. [You may assume without proof that every permutation is the product of disjoint cycles.]

(a) Define the *sign* of a permutation in S_n, and prove that it is well defined. Define the *alternating group* A_n.

(b) Show that S_n is generated by the set $\{(1\ 2), (1\ 2\ 3\ \ldots\ n)\}$.

Given $1 \leqslant k < n$, prove that the set $\{(1\ 1{+}k), (1\ 2\ 3\ \ldots\ n)\}$ generates S_n if and only if k and n are coprime.

Paper 4, Section I
1D Numbers and Sets

(a) Show that for all positive integers z and n, either $z^{2n} \equiv 0 \pmod 3$ or $z^{2n} \equiv 1 \pmod 3$.

(b) If the positive integers x, y, z satisfy $x^2 + y^2 = z^2$, show that at least one of x and y must be divisible by 3. Can both x and y be odd?

Paper 4, Section I
2D Numbers and Sets

(a) Give the definitions of *relation* and *equivalence relation* on a set S.

(b) Let Σ be the set of ordered pairs (A, f) where A is a non-empty subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$. Let $\mathcal{R}$ be the relation on Σ defined by requiring $(A, f)\,\mathcal{R}\,(B, g)$ if the following two conditions hold:

(i) $(A \setminus B) \cup (B \setminus A)$ is finite and

(ii) there is a finite set $F \subset A \cap B$ such that $f(x) = g(x)$ for all $x \in A \cap B \setminus F$.

Show that $\mathcal{R}$ is an equivalence relation on Σ.

Paper 4, Section II
5D Numbers and Sets

(a) State and prove the Fermat–Euler Theorem. Deduce Fermat's Little Theorem. State Wilson's Theorem.

(b) Let p be an odd prime. Prove that $X^2 \equiv -1 \pmod{p}$ is solvable if and only if $p \equiv 1 \pmod{4}$.

(c) Let p be prime. If h and k are non-negative integers with $h + k = p - 1$, prove that $h!k! + (-1)^h \equiv 0 \pmod{p}$.

Paper 4, Section II
6D Numbers and Sets

(a) Define what it means for a set to be *countable*.

(b) Let A be an infinite subset of the set of natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$. Prove that there is a bijection $f : \mathbb{N} \to A$.

(c) Let A_n be the set of natural numbers whose decimal representation ends with exactly $n - 1$ zeros. For example, $71 \in A_1$, $70 \in A_2$ and $15000 \in A_4$. By applying the result of part (b) with $A = A_n$, construct a bijection $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Deduce that the set of rationals is countable.

(d) Let A be an infinite set of positive real numbers. If every sequence $(a_j)_{j=1}^{\infty}$ of distinct elements with $a_j \in A$ for each j has the property that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} a_j = 0,$$

prove that A is countable.

[*You may assume without proof that a countable union of countable sets is countable.*]

Paper 4, Section II

7D Numbers and Sets

(a) For positive integers n, m, k with $k \leqslant n$, show that

$$\binom{n}{k}\left(\frac{k}{n}\right)^m = \binom{n-1}{k-1}\sum_{\ell=0}^{m-1} a_{n,m,\ell}\left(\frac{k-1}{n-1}\right)^{m-1-\ell}$$

giving an explicit formula for $a_{n,m,\ell}$. [You may wish to consider the expansion of $\left(\frac{k-1}{n-1}+\frac{1}{n-1}\right)^{m-1}$.]

(b) For a function $f : [0,1] \to \mathbb{R}$ and each integer $n \geqslant 1$, the function $B_n(f) : [0,1] \to \mathbb{R}$ is defined by

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

For any integer $m \geqslant 0$ let $f_m(x) = x^m$. Show that $B_n(f_0)(x) = 1$ and $B_n(f_1)(x) = x$ for all $n \geqslant 1$ and $x \in [0,1]$.

Show that for each integer $m \geqslant 0$ and each $x \in [0,1]$,

$$B_n(f_m)(x) \to f_m(x) \quad \text{as} \quad n \to \infty\,.$$

Deduce that for each integer $m \geqslant 0$,

$$\lim_{n\to\infty} \frac{1}{4^n}\sum_{k=0}^{2n}\left(\frac{k}{n}\right)^m\binom{2n}{k} = 1.$$

Paper 4, Section II

8D Numbers and Sets

Let $(a_k)_{k=1}^{\infty}$ be a sequence of real numbers.

(a) Define what it means for $(a_k)_{k=1}^{\infty}$ to converge. Define what it means for the series $\sum_{k=1}^{\infty} a_k$ to converge.

Show that if $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k)_{k=1}^{\infty}$ converges to 0.

If $(a_k)_{k=1}^{\infty}$ converges to $a \in \mathbb{R}$, show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = a\,.$$

(b) Suppose $a_k > 0$ for every k. Let $u_n = \sum_{k=1}^{n} \left(a_k + \frac{1}{a_k}\right)$ and $v_n = \sum_{k=1}^{n} \left(a_k - \frac{1}{a_k}\right)$.

Show that $(u_n)_{n=1}^{\infty}$ does not converge.

Give an example of a sequence $(a_k)_{k=1}^{\infty}$ with $a_k > 0$ and $a_k \neq 1$ for every k such that $(v_n)_{n=1}^{\infty}$ converges.

If $(v_n)_{n=1}^{\infty}$ converges, show that $\dfrac{u_n}{n} \to 2$.

Paper 2, Section I

3F Probability

Let X be a non-negative integer-valued random variable such that $0 < \mathbb{E}(X^2) < \infty$. Prove that

$$\frac{\mathbb{E}(X)^2}{\mathbb{E}(X^2)} \leqslant \mathbb{P}(X > 0) \leqslant \mathbb{E}(X).$$

[You may use any standard inequality.]

Paper 2, Section I

4F Probability

Let X and Y be real-valued random variables with joint density function

$$f(x,y) = \begin{cases} xe^{-x(y+1)} & \text{if } x \geqslant 0 \text{ and } y \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the conditional probability density function of Y given X.

(ii) Find the expectation of Y given X.

Paper 2, Section II

9F Probability

For a positive integer N, $p \in [0,1]$, and $k \in \{0,1,\ldots,N\}$, let

$$p_k(N,p) = \binom{N}{k} p^k (1-p)^{N-k}.$$

(a) For fixed N and p, show that $p_k(N,p)$ is a probability mass function on $\{0,1,\ldots,N\}$ and that the corresponding probability distribution has mean Np and variance $Np(1-p)$.

(b) Let $\lambda > 0$. Show that, for any $k \in \{0,1,2,\ldots\}$,

$$\lim_{N\to\infty} p_k(N,\lambda/N) = \frac{e^{-\lambda}\lambda^k}{k!}. \qquad (*)$$

Show that the right-hand side of $(*)$ is a probability mass function on $\{0,1,2,\ldots\}$.

(c) Let $p \in (0,1)$ and let $a,b \in \mathbb{R}$ with $a < b$. For all N, find integers $k_a(N)$ and $k_b(N)$ such that

$$\sum_{k=k_a(N)}^{k_b(N)} p_k(N,p) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}x^2}\,dx \qquad \text{as } N \to \infty.$$

[You may use the Central Limit Theorem.]

Paper 2, Section II

10F Probability

(a) For any random variable X and $\lambda > 0$ and $t > 0$, show that

$$\mathbb{P}(X > t) \leqslant \mathbb{E}(e^{\lambda X})e^{-\lambda t}.$$

For a standard normal random variable X, compute $\mathbb{E}(e^{\lambda X})$ and deduce that

$$\mathbb{P}(X > t) \leqslant e^{-\frac{1}{2}t^2}.$$

(b) Let $\mu, \lambda > 0$, $\mu \neq \lambda$. For independent random variables X and Y with distributions $\mathrm{Exp}(\lambda)$ and $\mathrm{Exp}(\mu)$, respectively, compute the probability density functions of $X + Y$ and $\min\{X, Y\}$.

Paper 2, Section II

11F Probability

Let $\beta > 0$. The *Curie–Weiss Model* of ferromagnetism is the probability distribution defined as follows. For $n \in \mathbb{N}$, define random variables $S_1, \ldots, S_n$ with values in $\{\pm 1\}$ such that the probabilities are given by

$$\mathbb{P}(S_1 = s_1, \ldots, S_n = s_n) = \frac{1}{Z_{n,\beta}} \exp\left(\frac{\beta}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j \right)$$

where $Z_{n,\beta}$ is the normalisation constant

$$Z_{n,\beta} = \sum_{s_1 \in \{\pm 1\}} \cdots \sum_{s_n \in \{\pm 1\}} \exp\left(\frac{\beta}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j \right).$$

(a) Show that $\mathbb{E}(S_i) = 0$ for any i.

(b) Show that $\mathbb{P}(S_2 = +1 | S_1 = +1) \geqslant \mathbb{P}(S_2 = +1)$. [You may use $\mathbb{E}(S_i S_j) \geqslant 0$ for all i, j without proof.]

(c) Let $M = \frac{1}{n} \sum_{i=1}^{n} S_i$. Show that M takes values in $E_n = \{-1 + \frac{2k}{n} : k = 0, \ldots, n\}$, and that for each $m \in E_n$ the number of possible values of $(S_1, \ldots, S_n)$ such that $M = m$ is

$$\frac{n!}{\left(\frac{1+m}{2}n\right)! \left(\frac{1-m}{2}n\right)!}.$$

Find $\mathbb{P}(M = m)$ for any $m \in E_n$.

Paper 2, Section II
12F Probability

(a) Let $k \in \{1, 2, \dots\}$. For $j \in \{0, \dots, k+1\}$, let D_j be the first time at which a simple symmetric random walk on $\mathbb{Z}$ with initial position j at time 0 hits 0 or $k+1$. Show $\mathbb{E}(D_j) = j(k+1-j)$. [If you use a recursion relation, you do not need to prove that its solution is unique.]

(b) Let (S_n) be a simple symmetric random walk on $\mathbb{Z}$ starting at 0 at time $n = 0$. For $k \in \{1, 2, \dots\}$, let T_k be the first time at which (S_n) has visited k distinct vertices. In particular, $T_1 = 0$. Show $\mathbb{E}(T_{k+1} - T_k) = k$ for $k \geqslant 1$. [You may use without proof that, conditional on $S_{T_k} = i$, the random variables $(S_{T_k+n})_{n \geqslant 0}$ have the distribution of a simple symmetric random walk starting at i.]

(c) For $n \geqslant 3$, let $\mathbb{Z}_n$ be the circle graph consisting of vertices $0, \dots, n-1$ and edges between k and $k+1$ where n is identified with 0. Let (Y_i) be a simple random walk on $\mathbb{Z}_n$ starting at time 0 from 0. Thus $Y_0 = 0$ and conditional on Y_i the random variable Y_{i+1} is $Y_i \pm 1$ with equal probability (identifying $k+n$ with k).

The *cover time* T of the simple random walk on $\mathbb{Z}_n$ is the first time at which the random walk has visited all vertices. Show that $\mathbb{E}(T) = n(n-1)/2$.

Paper 3, Section I

3B Vector Calculus

Use the change of variables $x = r\cosh\theta$, $y = r\sinh\theta$ to evaluate

$$\int_A y\,dx\,dy\,,$$

where A is the region of the xy-plane bounded by the two line segments:

$y = 0, \quad 0 \leqslant x \leqslant 1\,;$

$5y = 3x, \quad 0 \leqslant x \leqslant \frac{5}{4}\,;$

and the curve

$x^2 - y^2 = 1, \quad x \geqslant 1\,.$

Paper 3, Section I

4B Vector Calculus

(a) The two sets of basis vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$ (where $i = 1, 2, 3$) are related by

$$\mathbf{e}'_i = R_{ij}\mathbf{e}_j\,,$$

where R_{ij} are the entries of a rotation matrix. The components of a vector $\mathbf{v}$ with respect to the two bases are given by

$$\mathbf{v} = v_i\mathbf{e}_i = v'_i\mathbf{e}'_i\,.$$

Derive the relationship between v_i and v'_i.

(b) Let $\mathbf{T}$ be a 3×3 array defined in each (right-handed orthonormal) basis. Using part (a), state and prove the quotient theorem as applied to $\mathbf{T}$.

Paper 3, Section II
9B Vector Calculus

(a) The time-dependent vector field $\mathbf{F}$ is related to the vector field $\mathbf{B}$ by

$$\mathbf{F}(\mathbf{x},t) = \mathbf{B}(\mathbf{z})\,,$$

where $\mathbf{z} = t\mathbf{x}$. Show that

$$(\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{F} = t\,\frac{\partial \mathbf{F}}{\partial t}\,.$$

(b) The vector fields $\mathbf{B}$ and $\mathbf{A}$ satisfy $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Show that $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$.

(c) The vector field $\mathbf{B}$ satisfies $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. Show that

$$\mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla}\times\big(\mathbf{D}(\mathbf{x})\times\mathbf{x}\big)\,,$$

where

$$\mathbf{D}(\mathbf{x}) = \int_0^1 t\,\mathbf{B}(t\mathbf{x})\,dt\,.$$

Paper 3, Section II
10B Vector Calculus

By a suitable choice of $\mathbf{u}$ in the divergence theorem

$$\int_V \boldsymbol{\nabla}\cdot\mathbf{u}\,dV = \int_S \mathbf{u}\cdot d\mathbf{S}\,,$$

show that

$$\int_V \boldsymbol{\nabla}\phi\,dV = \int_S \phi\,d\mathbf{S} \qquad (*)$$

for any continuously differentiable function ϕ.

For the curved surface of the cone

$$\mathbf{x} = (r\cos\theta, r\sin\theta,\ \sqrt{3}\,r), \qquad 0 \leqslant \sqrt{3}\,r \leqslant 1,\quad 0 \leqslant \theta \leqslant 2\pi,$$

show that $d\mathbf{S} = (\sqrt{3}\,\cos\theta, \sqrt{3}\,\sin\theta, -1)\,r\,dr\,d\theta$.

Verify that $(*)$ holds for this cone and $\phi(x,y,z) = z^2$.

Paper 3, Section II

11B Vector Calculus

(a) Let $\mathbf{x} = \mathbf{r}(s)$ be a smooth curve parametrised by arc length s. Explain the meaning of the terms in the equation

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n}\,,$$

where $\kappa(s)$ is the curvature of the curve.

Now let $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. Show that there is a scalar $\tau(s)$ (the torsion) such that

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}$$

and derive an expression involving κ and τ for $\dfrac{d\mathbf{n}}{ds}$.

(b) Given a (nowhere zero) vector field $\mathbf{F}$, the *field lines*, or *integral curves*, of $\mathbf{F}$ are the curves parallel to $\mathbf{F}(\mathbf{x})$ at each point $\mathbf{x}$. Show that the curvature κ of the field lines of $\mathbf{F}$ satisfies

$$\frac{\mathbf{F} \times (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{F}}{F^3} = \pm\kappa\mathbf{b}\,, \qquad (*)$$

where $F = |\mathbf{F}|$.

(c) Use $(*)$ to find an expression for the curvature at the point (x, y, z) of the field lines of $\mathbf{F}(x, y, z) = (x, y, -z)$.

Paper 3, Section II

12B Vector Calculus

Let S be a piecewise smooth closed surface in $\mathbb{R}^3$ which is the boundary of a volume V.

(a) The smooth functions ϕ and ϕ_1 defined on $\mathbb{R}^3$ satisfy

$$\nabla^2\phi = \nabla^2\phi_1 = 0$$

in V and $\phi(\mathbf{x}) = \phi_1(\mathbf{x}) = f(\mathbf{x})$ on S. By considering an integral of $\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\psi$, where $\psi = \phi - \phi_1$, show that $\phi_1 = \phi$.

(b) The smooth function u defined on $\mathbb{R}^3$ satisfies $u(\mathbf{x}) = f(\mathbf{x}) + C$ on S, where f is the function in part (a) and C is constant. Show that

$$\int_V \boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}u\, dV \geqslant \int_V \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi\, dV$$

where ϕ is the function in part (a). When does equality hold?

(c) The smooth function $w(\mathbf{x}, t)$ satisfies

$$\nabla^2 w = \frac{\partial w}{\partial t}$$

in V and $\dfrac{\partial w}{\partial t} = 0$ on S for all t. Show that

$$\frac{d}{dt}\int_V \boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}w\, dV \leqslant 0$$

with equality only if $\nabla^2 w = 0$ in V.

Paper 1, Section I

1A Vectors and Matrices

Consider $z \in \mathbb{C}$ with $|z| = 1$ and $\arg z = \theta$, where $\theta \in [0, \pi)$.

(a) Prove algebraically that the modulus of $1 + z$ is $2\cos\frac{1}{2}\theta$ and that the argument is $\frac{1}{2}\theta$. Obtain these results geometrically using the Argand diagram.

(b) Obtain corresponding results algebraically and geometrically for $1 - z$.

Paper 1, Section I

2C Vectors and Matrices

Let A and B be real $n \times n$ matrices.

Show that $(AB)^T = B^T A^T$.

For any square matrix, the *matrix exponential* is defined by the series

$$e^A = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} .$$

Show that $(e^A)^T = e^{A^T}$. [You are not required to consider issues of convergence.]

Calculate, in terms of A and A^T, the matrices Q_0, Q_1 and Q_2 in the series for the matrix product

$$e^{tA}\, e^{tA^T} = \sum_{k=0}^{\infty} Q_k t^k , \quad \text{where} \quad t \in \mathbb{R}.$$

Hence obtain a relation between A and A^T which necessarily holds if e^{tA} is an orthogonal matrix.

Paper 1, Section II

5A Vectors and Matrices

(a) Define the *vector product* $\mathbf{x} \times \mathbf{y}$ of the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$. Use suffix notation to prove that

$$\mathbf{x} \times (\mathbf{x} \times \mathbf{y}) = \mathbf{x}\,(\mathbf{x} \cdot \mathbf{y}) - \mathbf{y}\,(\mathbf{x} \cdot \mathbf{x}).$$

(b) The vectors $\mathbf{x}_{n+1}$ $(n = 0, 1, 2, \ldots)$ are defined by $\mathbf{x}_{n+1} = \lambda\,\mathbf{a} \times \mathbf{x}_n$, where $\mathbf{a}$ and $\mathbf{x}_0$ are fixed vectors with $|\mathbf{a}| = 1$ and $\mathbf{a} \times \mathbf{x}_0 \neq \mathbf{0}$, and λ is a positive constant.

(i) Write $\mathbf{x}_2$ as a linear combination of $\mathbf{a}$ and $\mathbf{x}_0$. Further, for $n \geqslant 1$, express $\mathbf{x}_{n+2}$ in terms of λ and $\mathbf{x}_n$. Show, for $n \geqslant 1$, that $|\mathbf{x}_n| = \lambda^n\,|\mathbf{a} \times \mathbf{x}_0|$.

(ii) Let X_n be the point with position vector $\mathbf{x}_n$ $(n = 0, 1, 2, \ldots)$. Show that $X_1, X_2, \ldots$ lie on a pair of straight lines.

(iii) Show that the line segment X_nX_{n+1} $(n \geqslant 1)$ is perpendicular to $X_{n+1}X_{n+2}$. Deduce that X_nX_{n+1} is parallel to $X_{n+2}X_{n+3}$.

Show that $\mathbf{x}_n \to \mathbf{0}$ as $n \to \infty$ if $\lambda < 1$, and give a sketch to illustrate the case $\lambda = 1$.

(iv) The straight line through the points X_{n+1} and X_{n+2} makes an angle θ with the straight line through the points X_n and X_{n+3}. Find $\cos\theta$ in terms of λ.

Paper 1, Section II

6B Vectors and Matrices

(a) Show that the eigenvalues of any real $n \times n$ square matrix A are the same as the eigenvalues of A^T.

The eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the eigenvalues of A^TA are $\mu_1, \mu_2, \ldots, \mu_n$. Determine, by means of a proof or a counterexample, whether the following are necessary valid:

(i) $\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \lambda_i^2$;

(ii) $\prod_{i=1}^{n} \mu_i = \prod_{i=1}^{n} \lambda_i^2$.

(b) The 3×3 matrix B is given by

$$B = I + \mathbf{m}\mathbf{n}^T ,$$

where $\mathbf{m}$ and $\mathbf{n}$ are orthogonal real unit vectors and I is the 3×3 identity matrix.

(i) Show that $\mathbf{m} \times \mathbf{n}$ is an eigenvector of B, and write down a linearly independent eigenvector. Find the eigenvalues of B and determine whether B is diagonalisable.

(ii) Find the eigenvectors and eigenvalues of B^TB.

Paper 1, Section II
7B Vectors and Matrices

(a) Show that a square matrix A is anti-symmetric if and only if $\mathbf{x}^T A\mathbf{x} = 0$ for every vector $\mathbf{x}$.

(b) Let A be a real anti-symmetric $n \times n$ matrix. Show that the eigenvalues of A are imaginary or zero, and that the eigenvectors corresponding to distinct eigenvalues are orthogonal (in the sense that $\mathbf{x}^\dagger \mathbf{y} = 0$, where the dagger denotes the hermitian conjugate).

(c) Let A be a non-zero real 3×3 anti-symmetric matrix. Show that there is a real non-zero vector $\mathbf{a}$ such that $A\mathbf{a} = \mathbf{0}$.

Now let $\mathbf{b}$ be a real vector orthogonal to $\mathbf{a}$. Show that $A^2\mathbf{b} = -\theta^2\mathbf{b}$ for some real number θ.

The matrix e^A is defined by the exponential series $I + A + \frac{1}{2!}A^2 + \cdots$. Express $e^A\mathbf{a}$ and $e^A\mathbf{b}$ in terms of $\mathbf{a}, \mathbf{b}$, $A\mathbf{b}$ and θ.

[*You are not required to consider issues of convergence.*]

Paper 1, Section II
8C Vectors and Matrices

(a) Given $\mathbf{y} \in \mathbb{R}^3$ consider the linear transformation T which maps

$$\mathbf{x} \mapsto T\mathbf{x} = (\mathbf{x} \cdot \mathbf{e}_1)\, \mathbf{e}_1 + \mathbf{x} \times \mathbf{y}\,.$$

Express T as a matrix with respect to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and determine the rank and the dimension of the kernel of T for the cases (i) $\mathbf{y} = c_1\mathbf{e}_1$, where c_1 is a fixed number, and (ii) $\mathbf{y} \cdot \mathbf{e}_1 = 0$.

(b) Given that the equation

$$A\,B\,\mathbf{x} = \mathbf{d}\,,$$

where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & 1 \\ -3 & -2 & 1 \\ 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ k \end{pmatrix},$$

has a solution, show that $4k = 1$.