UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday, 3 June, 2016 9:00 am to 12:00 noon

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, . . . , ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Compute the continued fraction expansion of $\sqrt{14}$, and use it to find two solutions to $x^2 - 14y^2 = 2$ where x and y are positive integers.

2H Topics in Analysis

Let a_0, a_1, $a_2, \ldots$ be integers such that there exists an M with $M \geqslant |a_n|$ for all n. Show that, if infinitely many of the a_n are non-zero, then $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n!}$ is an irrational number.

3G Coding and Cryptography

Describe the Rabin–Williams scheme for coding a message x as x^2 modulo a certain N. Show that, if N is chosen appropriately, breaking this code is equivalent to factorising the product of two primes.

4F Automata and Formal Languages

(a) Construct a register machine to compute the function $f(m,n) := m+n$. State the relationship between partial recursive functions and partial computable functions. Show that the function $g(m,n) := mn$ is partial recursive.

(b) State Rice's theorem. Show that the set $A := \{n \in \mathbb{N} \mid |W_n| > 7\}$ is recursively enumerable but not recursive.

5K Statistical Modelling

(a) Let $Y_i = x_i^\intercal \beta + \varepsilon_i$ where ε_i for $i = 1, \ldots, n$ are independent and identically distributed. Let $Z_i = I(Y_i < 0)$ for $i = 1, \ldots, n$, and suppose that these variables follow a binary regression model with the complementary log-log link function $g(\mu) = \log(-\log(1-\mu))$. What is the probability density function of ε_1?

(b) The Newton–Raphson algorithm can be applied to compute the MLE, $\hat{\beta}$, in certain GLMs. Starting from $\beta^{(0)} = 0$, we let $\beta^{(t+1)}$ be the maximizer of the quadratic approximation of the log-likelihood $\ell(\beta; Y)$ around $\beta^{(t)}$:

$$\ell(\beta; Y) \approx \ell(\beta^{(t)}; Y) + (\beta - \beta^{(t)})^\intercal D\ell(\beta^{(t)}; Y) + (\beta - \beta^{(t)})^\intercal D^2\ell(\beta^{(t)}; Y)(\beta - \beta^{(t)}),$$

where $D\ell$ and $D^2\ell$ are the gradient and Hessian of the log-likelihood. What is the difference between this algorithm and Iterative Weighted Least Squares? Why might the latter be preferable?

6B Mathematical Biology

A stochastic birth–death process is given by the master equation

$$\frac{dp_n}{dt} = \lambda(p_{n-1} - p_n) + \mu\,[\,(n-1)p_{n-1} - np_n\,] + \beta\,[\,(n+1)p_{n+1} - np_n\,]\ ,$$

where $p_n(t)$ is the probability that there are n individuals in the population at time t for $n = 0, 1, 2, \ldots$ and $p_n = 0$ for $n < 0$. Give a brief interpretation of λ, μ and β.

Derive an equation for $\dfrac{\partial \phi}{\partial t}$, where ϕ is the generating function

$$\phi(s,t) = \sum_{n=0}^{\infty} s^n p_n(t)\,.$$

Now assume that $\beta > \mu$. Show that at steady state

$$\phi = \left(\frac{\beta - \mu}{\beta - \mu s}\right)^{\lambda/\mu}$$

and find the corresponding mean and variance.

7A Further Complex Methods

Consider the equation for $w(z)$:

$$w'' + p(z)w' + q(z)w = 0\,. \qquad (*)$$

State necessary and sufficient conditions on $p(z)$ and $q(z)$ for $z = 0$ to be (i) an *ordinary point* or (ii) a *regular singular point*. Derive the corresponding conditions for the point $z = \infty$.

Determine the most general equation of the form $(*)$ that has regular singular points at $z = 0$ and $z = \infty$, with all other points being ordinary.

8E Classical Dynamics

Using conservation of angular momentum $\mathbf{L} = L_a \mathbf{e}_a$ in the body frame, derive the Euler equations for a rigid body:

$$I_1\,\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\,\omega_3 = 0, \quad I_2\,\dot{\omega}_2 + (I_1 - I_3)\,\omega_3\,\omega_1 = 0, \quad I_3\,\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\,\omega_2 = 0.$$

[You may use the formula $\dot{\mathbf{e}}_a = \boldsymbol{\omega} \wedge \mathbf{e}_a$ without proof.]

Assume that the principal moments of inertia satisfy $I_1 < I_2 < I_3$. Determine whether a rotation about the principal 3-axis leads to stable or unstable perturbations.

9C Cosmology

The external gravitational potential $\Phi(r)$ due to a thin spherical shell of radius a and mass per unit area σ, centred at $r = 0$, will equal the gravitational potential due to a point mass M at $r = 0$, at any distance $r > a$, provided

$$\frac{Mr\Phi(r)}{2\pi\sigma a} + K(a)r = \int_{r-a}^{r+a} R\Phi(R)\, dR\,, \qquad (*)$$

where $K(a)$ depends on the radius of the shell. For which values of q does this equation have solutions of the form $\Phi(r) = Cr^q$, where C is constant? Evaluate $K(a)$ in each case and find the relation between the mass of the shell and M.

Hence show that the general gravitational force

$$F(r) = \frac{A}{r^2} + Br$$

has a potential satisfying $(*)$. What is the cosmological significance of the constant B?

SECTION II

10I Number Theory

(a) Define *Euler's totient function* $\phi(n)$ and show that $\sum_{d|n} \phi(d) = n$.

(b) State Lagrange's theorem concerning roots of polynomials mod p.

(c) Let p be a prime. Proving any results you need about primitive roots, show that $x^m \equiv 1 \pmod p$ has exactly $(m, p-1)$ roots.

(d) Show that if p and $3p-2$ are both primes then $N = p(3p-2)$ is a Fermat pseudoprime for precisely a third of all bases.

11H Topics in Analysis

Explain briefly how a positive irrational number x gives rise to a continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

with the a_j non-negative integers and $a_j \geqslant 1$ for $j \geqslant 1$.

Show that, if we write

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

for $n \geqslant 0$.

Use the observation [which need not be proved] that x lies between p_n/q_n and p_{n+1}/q_{n+1} to show that

$$|p_n/q_n - x| \leqslant 1/q_n q_{n+1}\,.$$

Show that $q_n \geqslant F_n$ where F_n is the nth Fibonacci number (thus $F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$), and conclude that

$$\left| \frac{p_n}{q_n} - x \right| \leqslant \frac{1}{F_n F_{n+1}}.$$

12K Statistical Modelling

For 31 days after the outbreak of the 2014 Ebola epidemic, the World Health Organization recorded the number of new cases per day in 60 hospitals in West Africa. Researchers are interested in modelling Y_{ij}, the number of new Ebola cases in hospital i on day $j \geqslant 2$, as a function of several covariates:

- `lab`: a Boolean factor for whether the hospital has laboratory facilities,
- `casesBefore`: number of cases at the hospital on the previous day,
- `urban`: a Boolean factor indicating an urban area,
- `country`: a factor with three categories, Guinea, Liberia, and Sierra Leone,
- `numDoctors`: number of doctors at the hospital,
- `tradBurials`: a Boolean factor indicating whether traditional burials are common in the region.

Consider the output of the following R code (with some lines omitted):

```
> fit.1 <- glm(newCases~lab+casesBefore+urban+country+numDoctors+tradBurials,
+ data=ebola,family=poisson)
> summary(fit.1)
Coefficients:
                     Estimate Std. Error z value Pr(>|z|)
(Intercept)          0.094731   0.050322   1.882   0.0598 .
labTRUE              0.011298   0.049498   0.228   0.8195
casesBefore          0.324744   0.007752  41.891  < 2e-16 ***
urbanTRUE           -0.091554   0.088212  -1.038   0.2993
countryLiberia       0.088490   0.034119   2.594   0.0095 **
countrySierra Leone -0.197474   0.036969  -5.342 9.21e-08 ***
numDoctors          -0.020819   0.004658  -4.470 7.83e-06 ***
tradBurialsTRUE      0.054296   0.031676   1.714   0.0865 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

(a) Would you conclude based on the z-tests that an urban setting does not affect the rate of infection?

(b) Explain how you would predict the total number of new cases that the researchers will record in Sierra Leone on day 32.

We fit a new model which includes an interaction term, and compute a test statistic using the code:

```
> fit.2 <- glm(newCases~casesBefore+country+country:casesBefore+numDoctors,
+ data=ebola,family=poisson)
> fit.2$deviance - fit.1$deviance
[1] 3.016138
```

(c) What is the distribution of the statistic computed in the last line?

(d) Under what conditions is the deviance of each model approximately chi-squared?

13B Mathematical Biology

The population densities of two types of cell are given by $U(x,t)$ and $V(x,t)$. The system is described by the equations

$$\begin{aligned} \frac{\partial U}{\partial t} &= \alpha\, U(1-U) + \chi\, \frac{\partial}{\partial x}\left(U\,\frac{\partial V}{\partial x}\right) + D\frac{\partial^2 U}{\partial x^2}\,, \\ \frac{\partial V}{\partial t} &= V(1-V) - \beta\, UV + \frac{\partial^2 V}{\partial x^2}\,, \end{aligned}$$

where α, β, χ and D are positive constants.

(a) Identify the terms which involve interaction between the cell types, and briefly describe what each of these terms might represent.

(b) Consider the system without spatial dynamics. Find the condition on β for there to be a non-trivial spatially homogeneous solution that is stable to spatially invariant disturbances.

(c) Consider now the full spatial system, and consider small spatial perturbations proportional to $\cos(kx)$ of the solution found in part (b). Show that for sufficiently large χ (the precise threshold should be found) the spatially homogeneous solution is stable to perturbations with either small or large wavenumber, but is unstable to perturbations at some intermediate wavenumber.

14E Classical Dynamics

A particle of unit mass is attached to one end of a light, stiff rod of length ℓ. The other end of the rod is held at a fixed position, such that the rod is free to swing in any direction. Write down the Lagrangian for the system giving a clear definition of any angular variables you introduce. [You should assume the acceleration g is constant.]

Find two independent constants of the motion.

The particle is projected horizontally with speed v from a point where the rod lies at an angle α to the downward vertical, with $0 < \alpha < \pi/2$. In terms of ℓ, g and α, find the critical speed v_c such that the particle always remains at its initial height.

The particle is now projected horizontally with speed v_c but from a point at angle $\alpha + \delta\alpha$ to the vertical, where $\delta\alpha/\alpha \ll 1$. Show that the height of the particle oscillates, and find the period of oscillation in terms of ℓ, g and α.

15F Logic and Set Theory

(a) State Zorn's Lemma, and use it to prove that every nontrivial distributive lattice L admits a lattice homomorphism $L \to \{0, 1\}$.

(b) Let S be a propositional theory in a given language $\mathcal{L}$. Sketch the way in which the equivalence classes of formulae of $\mathcal{L}$, modulo S-provable equivalence, may be made into a Boolean algebra. [Detailed proofs are not required, but you should define the equivalence relation explicitly.]

(c) Hence show how the Completeness Theorem for propositional logic may be deduced from the result of part (a).

16G Graph Theory

State Menger's theorem in both the vertex form and the edge form. Explain briefly how the edge form of Menger's theorem may be deduced from the vertex form.

(a) Show that if G is 3-connected then G contains a cycle of even length.

(b) Let G be a connected graph with all degrees even. Prove that $\lambda(G)$ is even. [*Hint: if S is a minimal set of edges whose removal disconnects G, let H be a component of $G - S$ and consider the degrees of the vertices of H in the graph $G - S$.*] Give an example to show that $\kappa(G)$ can be odd.

17H Galois Theory

(a) Let $f = t^5 - 9t + 3 \in \mathbb{Q}[t]$ and let L be the splitting field of f over $\mathbb{Q}$. Show that $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to S_5. Let α be a root of f. Show that $\mathbb{Q} \subseteq \mathbb{Q}(\alpha)$ is neither a radical extension nor a solvable extension.

(b) Let $f = t^{26} + 2$ and let L be the splitting field of f over $\mathbb{Q}$. Is it true that $\mathrm{Gal}(L/\mathbb{Q})$ has an element of order 29? Justify your answer. Using reduction mod p techniques, or otherwise, show that $\mathrm{Gal}(L/\mathbb{Q})$ has an element of order 3.

[*Standard results from the course may be used provided they are clearly stated.*]

18I Representation Theory

Let N be a proper normal subgroup of a finite group G and let U be an irreducible complex representation of G. Show that either U restricted to N is a sum of copies of a single irreducible representation of N, or else U is induced from an irreducible representation of some proper subgroup of G.

Recall that a p-group is a group whose order is a power of the prime number p. Deduce, by induction on the order of the group, or otherwise, that every irreducible complex representation of a p-group is induced from a 1-dimensional representation of some subgroup.

[You may assume that a non-abelian p-group G has an abelian normal subgroup which is not contained in the centre of G.]

19F Number Fields

Let K be a number field, and p a prime in $\mathbb{Z}$. Explain what it means for p to be *inert*, to *split completely*, and to be *ramified* in K.

(a) Show that if $[K:\mathbb{Q}] > 2$ and $\mathcal{O}_K = \mathbb{Z}[\alpha]$ for some $\alpha \in K$, then 2 does not split completely in K.

(b) Let $K = \mathbb{Q}(\sqrt{d})$, with $d \neq 0, 1$ and d square-free. Determine, in terms of d, whether $p = 2$ splits completely, is inert, or ramifies in K. Hence show that the primes which ramify in K are exactly those which divide D_K.

20G Algebraic Topology

Let $T = S^1 \times S^1$ be the 2-dimensional torus, and let X be constructed from T by removing a small open disc.

(a) Show that X is homotopy equivalent to $S^1 \vee S^1$.

(b) Show that the universal cover of X is homotopy equivalent to a tree.

(c) Exhibit (finite) cell complexes X, Y, such that X and Y are not homotopy equivalent but their universal covers $\widetilde{X}, \widetilde{Y}$ are.

[*State carefully any results from the course that you use.*]

21I Linear Analysis

Let H be a complex Hilbert space.

(a) Let $T : H \to H$ be a bounded linear map. Show that the spectrum of T is a subset of $\{\lambda \in \mathbb{C} : |\lambda| \leqslant \|T\|_{\mathcal{B}(H)}\}$.

(b) Let $T : H \to H$ be a bounded self-adjoint linear map. For $\lambda, \mu \in \mathbb{C}$, let $E_\lambda := \{x \in H : Tx = \lambda x\}$ and $E_\mu := \{x \in H : Tx = \mu x\}$. If $\lambda \neq \mu$, show that $E_\lambda \perp E_\mu$.

(c) Let $T : H \to H$ be a compact self-adjoint linear map. For $\lambda \neq 0$, show that $E_\lambda := \{x \in H : Tx = \lambda x\}$ is finite-dimensional.

(d) Let $H_1 \subset H$ be a closed, proper, non-trivial subspace. Let P be the orthogonal projection to H_1.

(i) Prove that P is self-adjoint.

(ii) Determine the spectrum $\sigma(P)$ and the point spectrum $\sigma_p(P)$ of P.

(iii) Find a necessary and sufficient condition on H_1 for P to be compact.

22H Algebraic Geometry

(a) Let C be a smooth projective curve, and let D be an effective divisor on C. Explain how D defines a morphism ϕ_D from C to some projective space.

State a necessary and sufficient condition on D so that the pull-back of a hyperplane via ϕ_D is an element of the linear system $|D|$.

State necessary and sufficient conditions for ϕ_D to be an isomorphism onto its image.

(b) Let C now have genus 2, and let K be an effective canonical divisor. Show that the morphism ϕ_K is a morphism of degree 2 from C to $\mathbb{P}^1$.

Consider the divisor $K + P_1 + P_2$ for points P_i with $P_1 + P_2 \not\sim K$. Show that the linear system associated to this divisor induces a morphism ϕ from C to a quartic curve in $\mathbb{P}^2$. Show furthermore that $\phi(P) = \phi(Q)$, with $P \neq Q$, if and only if $\{P, Q\} = \{P_1, P_2\}$.

[*You may assume the Riemann–Roch theorem.*]

23G Differential Geometry

For $S \subset \mathbf{R}^3$ a smooth embedded surface, define what is meant by a *geodesic curve* on S. Show that any geodesic curve $\gamma(t)$ has constant speed $|\dot{\gamma}(t)|$.

For any point $P \in S$, show that there is a parametrization $\phi : U \to V$ of some open neighbourhood V of P in S, with $U \subset \mathbf{R}^2$ having coordinates (u, v), for which the first fundamental form is

$$du^2 + G(u,v)dv^2,$$

for some strictly positive smooth function G on U. State a formula for the Gaussian curvature K of S in V in terms of G. If $K \equiv 0$ on V, show that G is a function of v only, and that we may reparametrize so that the metric is locally of the form $du^2 + dw^2$, for appropriate local coordinates (u, w).

[*You may assume that for any $P \in S$ and nonzero $\xi \in T_P S$, there exists (for some $\epsilon > 0$) a unique geodesic $\gamma : (-\epsilon, \epsilon) \to S$ with $\gamma(0) = P$ and $\dot{\gamma}(0) = \xi$, and that such geodesics depend smoothly on the initial conditions P and ξ.*]

24J Probability and Measure

Give the definitions of the *convolution* $f * g$ and of the *Fourier transform* $\widehat{f}$ of f, and show that $\widehat{f * g} = \widehat{f}\,\widehat{g}$. State what it means for Fourier inversion to hold for a function f.

State the Plancherel identity and compute the L^2 norm of the Fourier transform of the function $f(x) = e^{-x}\mathbf{1}_{[0,1]}$.

Suppose that (f_n), f are functions in L^1 such that $f_n \to f$ in L^1 as $n \to \infty$. Show that $\widehat{f_n} \to \widehat{f}$ uniformly.

Give the definition of weak convergence, and state and prove the Central Limit Theorem.

25J Applied Probability

(a) Give the definition of a *renewal process.* Let $(N_t)_{t\geqslant 0}$ be a renewal process associated with (ξ_i) with $\mathbb{E}\,\xi_1 = 1/\lambda < \infty$. Show that almost surely

$$\frac{N_t}{t} \to \lambda \quad \text{as } t \to \infty.$$

(b) Give the definition of *Kingman's n-coalescent.* Let τ be the first time that all blocks have coalesced. Find an expression for $\mathbb{E}\,e^{-q\tau}$. Let L_n be the total length of the branches of the tree, i.e., if τ_i is the first time there are i lineages, then $L_n = \sum_{i=2}^{n} i(\tau_{i-1} - \tau_i)$. Show that $\mathbb{E}\,L_n \sim 2\log n$ as $n \to \infty$. Show also that $\mathrm{Var}(L_n) \leqslant C$ for all n, where C is a positive constant, and that in probability

$$\frac{L_n}{\mathbb{E}\,L_n} \to 1 \quad \text{as } n \to \infty.$$

26J Principles of Statistics

Consider a decision problem with parameter space Θ. Define the concepts of a *Bayes decision rule* δ_π and of a *least favourable prior.*

Suppose π is a prior distribution on Θ such that the Bayes risk of the Bayes rule equals $\sup_{\theta\in\Theta} R(\delta_\pi, \theta)$, where $R(\delta, \theta)$ is the risk function associated to the decision problem. Prove that δ_π is least favourable.

Now consider a random variable X arising from the binomial distribution $Bin(n, \theta)$, where $\theta \in \Theta = [0, 1]$. Construct a least favourable prior for the squared risk $R(\delta, \theta) = E_\theta(\delta(X) - \theta)^2$. [You may use without proof the fact that the Bayes rule for quadratic risk is given by the posterior mean.]

27K Stochastic Financial Models

Let U be concave and strictly increasing, and let $\mathcal{A}$ be a vector space of random variables. For every random variable Z let

$$F(Z) = \sup_{X \in \mathcal{A}} \mathbb{E}[U(X + Z)]$$

and suppose there exists a random variable $X_Z \in \mathcal{A}$ such that

$$F(Z) = \mathbb{E}[U(X_Z + Z)].$$

For a random variable Y, let $\pi(Y)$ be such that $F(Y - \pi(Y)) = F(0)$.

(a) Show that for every constant a we have $\pi(Y + a) = \pi(Y) + a$, and that if $\mathbb{P}(Y_1 \leqslant Y_2) = 1$, then $\pi(Y_1) \leqslant \pi(Y_2)$. Hence show that if $\mathbb{P}(a \leqslant Y \leqslant b) = 1$ for constants $a \leqslant b$, then $a \leqslant \pi(Y) \leqslant b$.

(b) Show that $Y \mapsto \pi(Y)$ is concave, and hence show $t \mapsto \pi(tY)/t$ is decreasing for $t > 0$.

(c) Assuming U is continuously differentiable, show that $\pi(tY)/t$ converges as $t \to 0$, and that there exists a random variable X_0 such that

$$\lim_{t \to 0} \frac{\pi(tY)}{t} = \frac{\mathbb{E}[U'(X_0)Y]}{\mathbb{E}[U'(X_0)]}.$$

28K Optimization and Control

State transversality conditions that can be used with Pontryagin's maximum principle and say when they are helpful.

Given T, it is desired to maximize $c_1 x_1(T) + c_2 x_2(T)$, where

$$\begin{aligned} \dot{x}_1 &= u_1(a_1 x_1 + a_2 x_2), \\ \dot{x}_2 &= u_2(a_1 x_1 + a_2 x_2), \end{aligned}$$

and $u = (u_1, u_2)$ is a time-varying control such that $u_1 \geqslant 0$, $u_2 \geqslant 0$ and $u_1 + u_2 = 1$. Suppose that $x_1(0)$ and $x_2(0)$ are positive, and that $0 < a_2 < a_1$ and $0 < c_1 < c_2$. Find the optimal control at times close to T. Show that over $[0, T]$ the optimal control is constant, or makes exactly one switch, the latter happening if and only if

$$c_2 e^{a_2 T} < c_1 + \frac{a_1 c_2}{a_2}\left(e^{a_2 T} - 1\right).$$

29C Asymptotic Methods

Consider the equation

$$\epsilon^2 \frac{d^2y}{dx^2} = Q(x)y\,, \tag{1}$$

where $\epsilon > 0$ is a small parameter and $Q(x)$ is smooth. Search for solutions of the form

$$y(x) = \exp\left[\frac{1}{\epsilon}\Big(S_0(x) + \epsilon S_1(x) + \epsilon^2 S_2(x) + \cdots\Big)\right],$$

and, by equating powers of ϵ, obtain a collection of equations for the $\{S_j(x)\}_{j=0}^{\infty}$ which is formally equivalent to (1). By solving explicitly for S_0 and S_1 derive the Liouville–Green approximate solutions $y^{LG}(x)$ to (1).

For the case $Q(x) = -V(x)$, where $V(x) \geqslant V_0$ and V_0 is a positive constant, consider the eigenvalue problem

$$\frac{d^2y}{dx^2} + E\,V(x)y = 0\,, \qquad y(0) = y(\pi) = 0\,. \tag{2}$$

Show that any eigenvalue E is necessarily positive. Solve the eigenvalue problem exactly when $V(x) = V_0$.

Obtain Liouville–Green approximate eigenfunctions $y_n^{LG}(x)$ for (2) with $E \gg 1$, and give the corresponding Liouville–Green approximation to the eigenvalues E_n^{LG}. Compare your results to the exact eigenvalues and eigenfunctions in the case $V(x) = V_0$, and comment on this.

30E Dynamical Systems

Consider the map defined on $\mathbb{R}$ by

$$F(x) = \begin{cases} 3x & x \leqslant \frac{1}{2} \\ 3(1-x) & x \geqslant \frac{1}{2} \end{cases}$$

and let I be the open interval $(0,1)$. Explain what it means for F to have a *horseshoe* on I by identifying the relevant intervals in the definition.

Let $\Lambda = \{x : F^n(x) \in I, \forall n \geqslant 0\}$. Show that $F(\Lambda) = \Lambda$.

Find the sets $\Lambda_1 = \{x : F(x) \in I\}$ and $\Lambda_2 = \{x : F^2(x) \in I\}$.

Consider the ternary (base-3) representation $x = 0 \cdot x_1 x_2 x_3 \ldots$ of numbers in I. Show that

$$F(0 \cdot x_1 x_2 x_3 \ldots) = \begin{cases} x_1 \cdot x_2 x_3 x_4 \ldots & x \leqslant \frac{1}{2} \\ \sigma(x_1) \cdot \sigma(x_2)\sigma(x_3)\sigma(x_4) \ldots & x \geqslant \frac{1}{2} \end{cases},$$

where the function $\sigma(x_i)$ of the ternary digits should be identified. What is the ternary representation of the non-zero fixed point? What do the ternary representations of elements of Λ have in common?

Show that F has sensitive dependence on initial conditions on Λ, that F is topologically transitive on Λ, and that periodic points are dense in Λ. [*Hint: You may assume that* $F^n(0 \cdot x_1 \ldots x_{n-1} 0 x_{n+1} x_{n+2} \ldots) = 0 \cdot x_{n+1} x_{n+2} \ldots$ *for* $x \in \Lambda$.]

Briefly state the relevance of this example to the relationship between Glendinning's and Devaney's definitions of chaos.

31A Principles of Quantum Mechanics

(a) Consider a quantum system with Hamiltonian $H = H_0 + V$, where H_0 is independent of time. Define the interaction picture corresponding to this Hamiltonian and derive an expression for the time derivative of an operator in the interaction picture, assuming it is independent of time in the Schrödinger picture.

(b) The Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ satisfy

$$\sigma_i \, \sigma_j \;=\; \delta_{ij} \;+\; i \, \epsilon_{ijk} \, \sigma_k \; .$$

Explain briefly how these properties allow $\boldsymbol{\sigma}$ to be used to describe a quantum system with spin $\frac{1}{2}$.

(c) A particle with spin $\frac{1}{2}$ has position and momentum operators $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$. The unitary operator corresponding to a rotation through an angle θ about an axis $\mathbf{n}$ is $U = \exp(-i\,\theta\,\mathbf{n} \cdot \mathbf{J}/\hbar)$ where $\mathbf{J}$ is the total angular momentum. Check this statement by considering the effect of an infinitesimal rotation on $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$ and $\boldsymbol{\sigma}$.

(d) Suppose that the particle in part (c) has Hamiltonian $H = H_0 + V$ with

$$H_0 = \frac{1}{2m}\hat{\mathbf{p}}^2 \,+\, \alpha\,\mathbf{L} \cdot \boldsymbol{\sigma} \qquad \text{and} \qquad V = B\,\sigma_3\,,$$

where $\mathbf{L}$ is the orbital angular momentum and α, B are constants. Show that all components of $\mathbf{J}$ are independent of time in the interaction picture. Is this true in the Heisenberg picture?

[You may quote commutation relations of $\mathbf{L}$ with $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$.]

32A Applications of Quantum Mechanics

Let $\Lambda \subset \mathbb{R}^2$ be a Bravais lattice. Define the dual lattice Λ^* and show that

$$V(\mathbf{x}) \;=\; \sum_{\mathbf{q}\in\Lambda^*} V_{\mathbf{q}}\,\exp(i\mathbf{q}\cdot\mathbf{x})$$

obeys $V(\mathbf{x}+\boldsymbol{l}) = V(\mathbf{x})$ for all $\boldsymbol{l} \in \Lambda$, where $V_{\mathbf{q}}$ are constants. Suppose $V(\mathbf{x})$ is the potential for a particle of mass m moving in a two-dimensional crystal that contains a very large number of lattice sites of Λ and occupies an area $\mathcal{A}$. Adopting periodic boundary conditions, plane-wave states $|\,\mathbf{k}\,\rangle$ can be chosen such that

$$\langle\,\mathbf{x}\,|\,\mathbf{k}\,\rangle \;=\; \frac{1}{\mathcal{A}^{1/2}}\exp\left(i\mathbf{k}\cdot\mathbf{x}\right) \qquad \text{and} \qquad \langle\,\mathbf{k}\,|\,\mathbf{k}'\,\rangle \;=\; \delta_{\mathbf{k}\,\mathbf{k}'}\;.$$

The allowed wavevectors $\mathbf{k}$ are closely spaced and include all vectors in Λ^*. Find an expression for the matrix element $\langle\,\mathbf{k}\,|\,V(\mathbf{x})\,|\,\mathbf{k}'\,\rangle$ in terms of the coefficients $V_{\mathbf{q}}$. [You need not discuss additional details of the boundary conditions.]

Now suppose that $V(\mathbf{x}) = \lambda\,U(\mathbf{x})$, where $\lambda \ll 1$ is a dimensionless constant. Find the energy $E(\mathbf{k})$ for a particle with wavevector $\mathbf{k}$ to order λ^2 in non-degenerate perturbation theory. Show that this expansion in λ breaks down on the Bragg lines in $\mathbf{k}$-space defined by the condition

$$\mathbf{k}\cdot\mathbf{q} \;=\; \frac{1}{2}|\mathbf{q}|^2 \qquad \text{for} \qquad \mathbf{q}\in\Lambda^*\,,$$

and explain briefly, without additional calculations, the significance of this for energy levels in the crystal.

Consider the particular case in which Λ has primitive vectors

$$\mathbf{a}_1 \;=\; 2\pi\Big(\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}\Big)\;, \qquad \mathbf{a}_2 \;=\; 2\pi\,\frac{2}{\sqrt{3}}\mathbf{j}\;,$$

where $\mathbf{i}$ and $\mathbf{j}$ are orthogonal unit vectors. Determine the polygonal region in $\mathbf{k}$-space corresponding to the lowest allowed energy band.

33C Statistical Physics

(a) State the first law of thermodynamics. Derive the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V .$$

(b) Consider a thermodynamic system whose energy E at constant temperature T is volume independent, i.e.

$$\left(\frac{\partial E}{\partial V}\right)_T = 0 \,.$$

Show that this implies that the pressure has the form $p(T,V) = Tf(V)$ for some function f.

(c) For a photon gas inside a cavity of volume V, the energy E and pressure p are given in terms of the energy density U, which depends only on the temperature T, by

$$E(T,V) = U(T)V\,, \quad p(T,V) = \frac{1}{3}U(T)\,.$$

Show that this implies $U(T) = \sigma T^4$ where σ is a constant. Show that the entropy is

$$S = \frac{4}{3}\sigma T^3 V\,,$$

and calculate the energy $E(S,V)$ and free energy $F(T,V)$ in terms of their respective fundamental variables.

34E Electrodynamics

(a) A uniform, isotropic dielectric medium occupies the half-space $z > 0$. The region $z < 0$ is in vacuum. State the boundary conditions that should be imposed on $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{H}$ at $z = 0$.

(b) A linearly polarized electromagnetic plane wave, with magnetic field in the (x, y)-plane, is incident on the dielectric from $z < 0$. The wavevector $\mathbf{k}$ makes an acute angle θ_I to the normal $\hat{\mathbf{z}}$. If the dielectric has frequency-independent relative permittivity ϵ_r, show that the fraction of the incident power that is reflected is

$$\mathcal{R} = \left(\frac{n\cos\theta_I - \cos\theta_T}{n\cos\theta_I + \cos\theta_T}\right)^2 ,$$

where $n = \sqrt{\epsilon_r}$, and the angle θ_T should be specified. [You should ignore any magnetic response of the dielectric.]

(c) Now suppose that the dielectric moves at speed βc along the x-axis, the incident angle $\theta_I = 0$, and the magnetic field of the incident radiation is along the y-direction. Show that the reflected radiation propagates normal to the surface $z = 0$, has the same frequency as the incident radiation, and has magnetic field also along the y-direction. [*Hint: You may assume that under a standard Lorentz boost with speed $v = \beta c$ along the x-direction, the electric and magnetic field components transform as*

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} E_x \\ \gamma(E_y - vB_z) \\ \gamma(E_z + vB_y) \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} B_x \\ \gamma(B_y + vE_z/c^2) \\ \gamma(B_z - vE_y/c^2) \end{pmatrix} ,$$

where $\gamma = (1-\beta^2)^{-1/2}$.]

(d) Show that the fraction of the incident power reflected from the moving dielectric is

$$\mathcal{R}_\beta = \left(\frac{n/\gamma - \sqrt{1-\beta^2/n^2}}{n/\gamma + \sqrt{1-\beta^2/n^2}}\right)^2 .$$

35D General Relativity

A spherically symmetric static spacetime has metric

$$ds^2 = -\left(1 + r^2/b^2\right) dt^2 + \frac{dr^2}{1 + r^2/b^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

where $-\infty < t < \infty$, $r \geqslant 0$, b is a positive constant, and units such that $c = 1$ are used.

(a) Explain why a time-like geodesic may be assumed, without loss of generality, to lie in the equatorial plane $\theta = \pi/2$. For such a geodesic, show that the quantities

$$E = (1 + r^2/b^2)\,\dot{t} \qquad \text{and} \qquad h = r^2\dot{\phi}$$

are constants of the motion, where a dot denotes differentiation with respect to proper time, τ. Hence find a first-order differential equation for $r(\tau)$.

(b) Consider a massive particle fired from the origin, $r = 0$. Show that the particle will return to the origin and find the proper time taken.

(c) Show that circular orbits $r = a$ are possible for any $a > 0$ and determine whether such orbits are stable. Show that on any such orbit a clock measures coordinate time.

36B Fluid Dynamics II

A thin layer of fluid of viscosity μ occupies the gap between a rigid flat plate at $y=0$ and a flexible no-slip boundary at $y=h(x,t)$. The flat plate moves with constant velocity $U\mathbf{e}_x$ and the flexible boundary moves with no component of velocity in the x-direction.

State the two-dimensional lubrication equations governing the dynamics of the thin layer of fluid. Given a pressure gradient $\mathrm{d}p/\mathrm{d}x$, solve for the velocity profile $u(x,y,t)$ in the fluid and calculate the flux $q(x,t)$. Deduce that the pressure gradient satisfies

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\right)=\frac{\partial h}{\partial t}+\frac{U}{2}\frac{\partial h}{\partial x}.$$

The shape of the flexible boundary is a periodic travelling wave, i.e. $h(x,t)=h(x-ct)$ and $h(\xi+L)=h(\xi)$, where c and L are constants. There is no applied average pressure gradient, so the pressure is also periodic with $p(\xi+L)=p(\xi)$. Show that

$$\frac{\mathrm{d}p}{\mathrm{d}x}=6\mu\,(U-2c)\left(\frac{1}{h^2}-\frac{\langle h^{-2}\rangle}{\langle h^{-3}\rangle}\frac{1}{h^3}\right),$$

where $\langle ... \rangle = \dfrac{1}{L}\displaystyle\int_0^L ... \,\mathrm{d}x$ denotes the average over a period. Calculate the shear stress σ_{xy} on the plate.

The speed U is such that there is no need to apply an external tangential force to the plate in order to maintain its motion. Show that

$$U=6c\,\frac{\langle h^{-2}\rangle\langle h^{-2}\rangle-\langle h^{-1}\rangle\langle h^{-3}\rangle}{3\langle h^{-2}\rangle\langle h^{-2}\rangle-4\langle h^{-1}\rangle\langle h^{-3}\rangle}.$$

37D Waves

A duck swims at a constant velocity $(-V,0)$, where $V>0$, on the surface of infinitely deep water. Surface tension can be neglected, and the dispersion relation for the linear surface water waves (relative to fluid at rest) is $\omega^2 = g|\mathbf{k}|$. Show that the wavevector $\mathbf{k}$ of a plane harmonic wave that is steady in the duck's frame, i.e. of the form

$$\mathrm{Re}\left[A\, e^{i(k_1 x' + k_2 y)}\right] ,$$

where $x' = x + Vt$ and y are horizontal coordinates relative to the duck, satisfies

$$(k_1, k_2) = \frac{g}{V^2}\sqrt{p^2+1}\,(1,p)\,,$$

where $\hat{\mathbf{k}} = (\cos\phi, \sin\phi)$ and $p = \tan\phi$. [You may assume that $|\phi| < \pi/2$.]

Assume that the wave pattern behind the duck can be regarded as a Fourier superposition of such steady waves, i.e., the surface elevation η at $(x', y) = R(\cos\theta, \sin\theta)$ has the form

$$\eta = \mathrm{Re}\int_{-\infty}^{\infty} A(p)\, e^{i\lambda h(p;\theta)}\, \mathrm{d}p \quad \text{for } |\theta| < \tfrac{1}{2}\pi\,,$$

where

$$\lambda = \frac{gR}{V^2}\,, \quad h(p;\theta) = \sqrt{p^2+1}\,(\cos\theta + p\sin\theta)\ .$$

Show that, in the limit $\lambda \to \infty$ at fixed θ with $0 < \theta < \cot^{-1}(2\sqrt{2})$,

$$\eta \sim \sqrt{\frac{2\pi}{\lambda}}\ \mathrm{Re}\left\{ \frac{A(p_+)}{\sqrt{h_{pp}(p_+;\theta)}}\, e^{i\left(\lambda h(p_+;\theta) + \frac{1}{4}\pi\right)} \ +\ \frac{A(p_-)}{\sqrt{-h_{pp}(p_-;\theta)}}\, e^{i\left(\lambda h(p_-;\theta) - \frac{1}{4}\pi\right)} \right\},$$

where

$$p_\pm = -\tfrac{1}{4}\cot\theta \pm \tfrac{1}{4}\sqrt{\cot^2\theta - 8}$$

and h_{pp} denotes $\partial^2 h/\partial p^2$. Briefly interpret this result in terms of what is seen.

Without doing detailed calculations, briefly explain what is seen as $\lambda \to \infty$ at fixed θ with $\cot^{-1}(2\sqrt{2}) < \theta < \pi/2$. Very briefly comment on the case $\theta = \cot^{-1}(2\sqrt{2})$.

[*Hint: You may find the following results useful.*

$$h_p = \left\{p\cos\theta + (2p^2+1)\sin\theta\right\}(p^2+1)^{-1/2}\ ,$$

$$h_{pp} = (\cos\theta + 4p\sin\theta)\,(p^2+1)^{-1/2} - \left\{p\cos\theta + (2p^2+1)\sin\theta\right\} p(p^2+1)^{-3/2}\ .\ ]$$

38B Numerical Analysis

(a) Describe an implementation of the *power method* for determining the eigenvalue of largest modulus and its associated eigenvector for a matrix that has a unique eigenvalue of largest modulus.

Now let A be a real $n \times n$ matrix with distinct eigenvalues satisfying $|\lambda_n| = |\lambda_{n-1}|$ and $|\lambda_n| > |\lambda_i|$, $i = 1, \ldots, n-2$. The power method is applied to A, with an initial condition $\mathbf{x}^{(0)} = \sum_{i=1}^{n} c_i \mathbf{w}_i$ such that $c_{n-1}c_n \neq 0$, where $\mathbf{w}_i$ is the eigenvector associated with λ_i. Show that the power method does not converge. Explain why $\mathbf{x}^{(k)}$, $\mathbf{x}^{(k+1)}$ and $\mathbf{x}^{(k+2)}$ become linearly dependent as $k \to \infty$.

(b) Consider the following variant of the power method, called the two-stage power method, applied to the matrix A of part (a):

0. Pick $\mathbf{x}^{(0)} \in \mathbb{R}^n$ satisfying $\|\mathbf{x}^{(0)}\| = 1$. Let $0 < \varepsilon \ll 1$. Set $k = 0$ and $\mathbf{x}^{(1)} = A\mathbf{x}^{(0)}$.

1. Calculate $\mathbf{x}^{(k+2)} = A\mathbf{x}^{(k+1)}$ and calculate α, β that minimise

$$f(\alpha, \beta) = \|\mathbf{x}^{(k+2)} + \alpha\mathbf{x}^{(k+1)} + \beta\mathbf{x}^{(k)}\|.$$

2. If $f(\alpha, \beta) \leqslant \varepsilon$, solve $\lambda^2 + \alpha\lambda + \beta = 0$ and let the roots be λ_1 and λ_2. They are accepted as eigenvalues of A, and the corresponding eigenvectors are estimated as $\mathbf{x}^{(k+1)} - \lambda_2\mathbf{x}^{(k)}$ and $\mathbf{x}^{(k+1)} - \lambda_1\mathbf{x}^{(k)}$.

3. Otherwise, divide $\mathbf{x}^{(k+2)}$ and $\mathbf{x}^{(k+1)}$ by the current value of $\|\mathbf{x}^{(k+1)}\|$, increase k by 1 and return to Step **1.**

Explain the justification behind Step **2** of the algorithm.

(c) Let $n = 3$, and suppose that, for a large value of k, the two-stage power method yields the vectors

$$\mathbf{y}_k = \mathbf{x}^{(k)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_{k+1} = A\mathbf{x}^{(k)} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{y}_{k+2} = A^2\mathbf{x}^{(k)} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

Find two eigenvalues of A and the corresponding eigenvectors.

END OF PAPER