UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 2 June, 2016 9:00 am to 12:00 noon

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *. . . ,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Show that the exact power of a prime p dividing $N!$ is $\sum_{j=1}^{\infty} \lfloor \frac{N}{p^j} \rfloor$. By considering the prime factorisation of $\binom{2n}{n}$, show that

$$\frac{4^n}{2n+1} \leqslant \binom{2n}{n} \leqslant (2n)^{\pi(2n)}.$$

Setting $n = \lfloor \frac{x}{2} \rfloor$, deduce that for x sufficiently large

$$\pi(x) > \frac{\lfloor \frac{x}{2} \rfloor \log 3}{\log x} > \frac{x}{2 \log x}.$$

2H Topics in Analysis

In the game of 'Chicken', A and B drive fast cars directly at each other. If they both swerve, they both lose 10 status points; if neither swerves, they both lose 100 status points. If one swerves and the other does not, the swerver loses 20 status points and the non-swerver gains 40 status points. Find all the pairs of probabilistic strategies such that, if one driver maintains their strategy, it is not in the interest of the other to change theirs.

3G Coding and Cryptography

Describe in words the *unicity distance* of a cryptosystem.

Denote the cryptosystem by $\langle M, K, C \rangle$, in the usual way, and let $m \in M$ and $k \in K$ be random variables and $c = e(m, k)$. The unicity distance U is formally defined to be the least $n > 0$ such that $H(k|c^{(n)}) = 0$. Derive the formula

$$U = \frac{\log |K|}{\log |\Sigma| - H},$$

where $H = H(m)$, and Σ is the alphabet of the ciphertext. Make clear any assumptions you make.

The *redundancy* of a language is given by $R = 1 - \frac{H}{\log |\Sigma|}$. If a language has zero redundancy what is the unicity of any cryptosystem?

4F Automata and Formal Languages

(a) Define what it means for a context-free grammar (CFG) to be in *Chomsky normal form* (CNF). Can a CFG in CNF ever define a language containing ϵ? If G_{Chom} denotes the result of converting an arbitrary CFG G into one in CNF, state the relationship between $\mathcal{L}(G)$ and $\mathcal{L}(G_{\text{Chom}})$.

(b) Let G be a CFG in CNF, and let $w \in \mathcal{L}(G)$ be a word of length $|w| = n > 0$. Show that every derivation of w in G requires precisely $2n - 1$ steps. Using this, give an algorithm that, on input of any word v on the terminals of G, decides if $v \in \mathcal{L}(G)$ or not.

(c) Convert the following CFG G into a grammar in CNF:

$$S \rightarrow \; aSb \mid SS \mid \epsilon \,.$$

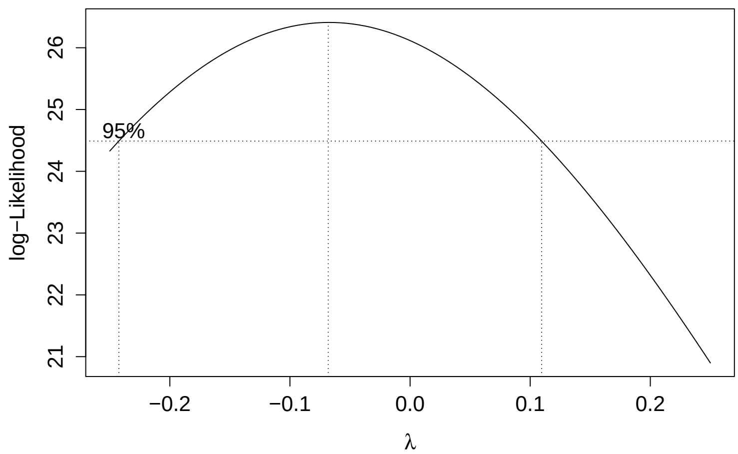
Does $\mathcal{L}(G) = \mathcal{L}(G_{\text{Chom}})$ in this case? Justify your answer.

5K Statistical Modelling

The R command

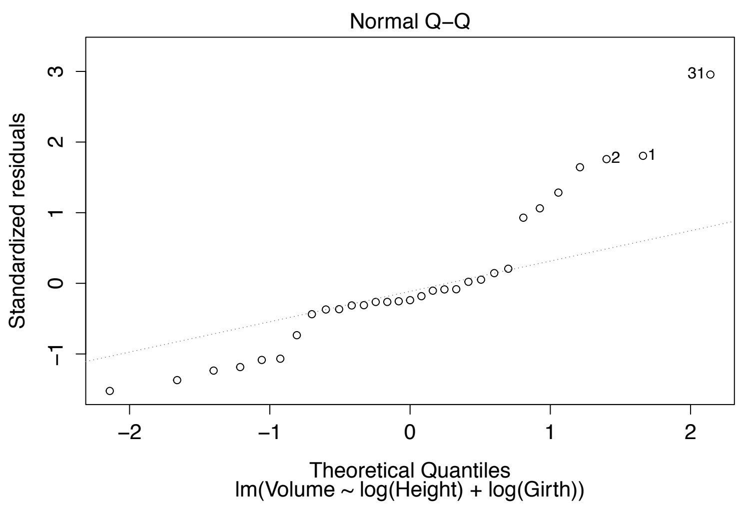
```
> boxcox(rainfall ~ month+elnino+month:elnino)
```

performs a Box–Cox transform of the response at several values of the parameter λ, and produces the following plot:



We fit two linear models and obtain the Q–Q plots for each fit, which are shown below in no particular order:

```
> fit.1 <- lm(rainfall ~ month+elnino+month:elnino)
> plot(fit.1,which=2)
> fit.2 <- lm(rainfall^-0.07 ~ month+elnino+month:elnino)
> plot(fit.2,which=2)
```



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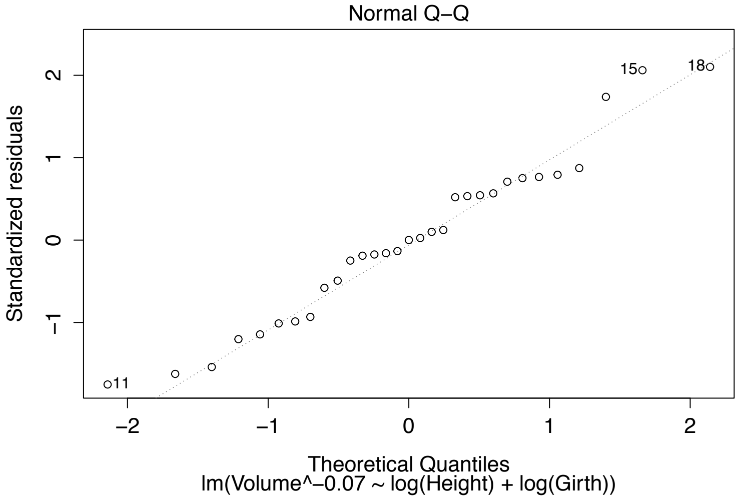
5K Statistical Modelling (continued)



Define the variable on the y-axis in the output of `boxcox`, and match each Q–Q plot to one of the models.

After choosing the model `fit.2`, the researcher calculates Cook's distance for the ith sample, which has high leverage, and compares it to the upper 0.01-point of an $F_{p,n-p}$ distribution, because the design matrix is of size $n \times p$. Provide an interpretation of this comparison in terms of confidence sets for $\hat{\beta}$. Is this confidence statement exact?

6B Mathematical Biology

A delay model for a population of size N_t at discrete time t is given by

$$N_{t+1} = \max\left\{ (r - N_{t-1}^2)N_t \, , \, 0 \right\} .$$

Show that for $r > 1$ there is a non-trivial equilibrium, and analyse its stability. Show that, as r is increased from 1, the equilibrium loses stability at $r = 3/2$ and find the approximate periodicity close to equilibrium at this point.

7A Further Complex Methods

The functions $f(x)$ and $g(x)$ have Laplace transforms $F(p)$ and $G(p)$ respectively, and $f(x) = g(x) = 0$ for $x \leqslant 0$. The *convolution* $h(x)$ of $f(x)$ and $g(x)$ is defined by

$$h(x) = \int_0^x f(y)\, g(x-y)\, dy \quad \text{for} \quad x > 0 \quad \text{and} \quad h(x) = 0 \quad \text{for} \quad x \leqslant 0\,.$$

Express the Laplace transform $H(p)$ of $h(x)$ in terms of $F(p)$ and $G(p)$.

Now suppose that $f(x) = x^\alpha$ and $g(x) = x^\beta$ for $x > 0$, where $\alpha,\, \beta > -1$. Find expressions for $F(p)$ and $G(p)$ by using a standard integral formula for the Gamma function. Find an expression for $h(x)$ by using a standard integral formula for the Beta function. Hence deduce that

$$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \mathrm{B}(z,w)$$

for all $\mathrm{Re}(z) > 0$, $\mathrm{Re}(w) > 0$.

8E Classical Dynamics

Consider a six-dimensional phase space with coordinates (q_i, p_i) for $i = 1, 2, 3$. Compute the Poisson brackets $\{L_i, L_j\}$, where $L_i = \epsilon_{ijk}\, q_j\, p_k$.

Consider the Hamiltonian

$$H = \frac{1}{2}\,|\mathbf{p}|^2 + V(\,|\mathbf{q}|\,)$$

and show that the resulting Hamiltonian system admits three Poisson-commuting independent first integrals.

9C Cosmology

A universe contains baryonic matter with background density $\rho_B(t)$ and density inhomogeneity $\delta_B(\mathbf{x},t)$, together with non-baryonic dark matter with background density $\rho_D(t)$ and density inhomogeneity $\delta_D(\mathbf{x},t)$. After the epoch of radiation–matter density equality, $t_{\rm eq}$, the background dynamics are governed by

$$H = \frac{2}{3t} \quad \text{and} \quad \rho_D = \frac{1}{6\pi G t^2}\,,$$

where H is the Hubble parameter.

The dark-matter density is much greater than the baryonic density ($\rho_D \gg \rho_B$) and so the time-evolution of the coupled density perturbations, at any point $\mathbf{x}$, is described by the equations

$$\begin{aligned} \ddot{\delta}_B + 2H\dot{\delta}_B &= 4\pi G\,\rho_D\,\delta_D\,, \\ \ddot{\delta}_D + 2H\dot{\delta}_D &= 4\pi G\,\rho_D\,\delta_D\,. \end{aligned}$$

Show that

$$\delta_D = \frac{\alpha}{t} + \beta\, t^{2/3}\,,$$

where α and β are independent of time. Neglecting modes in δ_D and δ_B that decay with increasing time, show that the baryonic density inhomogeneity approaches

$$\delta_B = \beta\, t^{2/3} + \gamma\,,$$

where γ is independent of time.

Briefly comment on the significance of your calculation for the growth of baryonic density inhomogeneities in the early universe.

SECTION II

10I Number Theory

What does it mean for a positive definite binary quadratic form to be *reduced*?

Prove that every positive definite binary quadratic form is equivalent to a reduced form, and that there are only finitely many reduced forms with given discriminant.

State a criterion for a positive integer n to be represented by a positive definite binary quadratic form with discriminant $d < 0$, and hence determine which primes p are represented by $x^2 + xy + 7y^2$.

11F Automata and Formal Languages

(a) Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite-state automaton. Define the *extended transition function* $\hat{\delta} : Q \times \Sigma^* \to Q$, and the *language accepted by* D, denoted $\mathcal{L}(D)$. Let $u, v \in \Sigma^*$, and $p \in Q$. Prove that $\hat{\delta}(p, uv) = \hat{\delta}(\hat{\delta}(p, u), v)$.

(b) Let $\sigma_1, \sigma_2, \ldots, \sigma_k \in \Sigma$ where $k \geqslant |Q|$, and let $p \in Q$.

(i) Show that there exist $0 \leqslant i < j \leqslant k$ such that $\hat{\delta}(p, \sigma_1 \cdots \sigma_i) = \hat{\delta}(p, \sigma_1 \cdots \sigma_j)$, where we interpret $\sigma_1 \cdots \sigma_i$ as ϵ if $i = 0$.

(ii) Show that $\hat{\delta}(p, \sigma_1 \cdots \sigma_i \sigma_{j+1} \cdots \sigma_k) = \hat{\delta}(p, \sigma_1 \cdots \sigma_k)$.

(iii) Show that $\hat{\delta}(p, \sigma_1 \cdots \sigma_i (\sigma_{i+1} \cdots \sigma_j)^t \sigma_{j+1} \cdots \sigma_k) = \hat{\delta}(p, \sigma_1 \cdots \sigma_k)$ for all $t \geqslant 1$.

(c) Prove the following version of the pumping lemma. Suppose $w \in \mathcal{L}(D)$, with $|w| \geqslant |Q|$. Then w can be broken up into three words $w = xyz$ such that $y \neq \epsilon$, $|xy| \leqslant |Q|$, and for all $t \geqslant 0$, the word $xy^t z$ is also in $\mathcal{L}(D)$.

(d) Hence show that

(i) if $\mathcal{L}(D)$ contains a word of length at least $|Q|$, then it contains infinitely many words;

(ii) if $\mathcal{L}(D) \neq \emptyset$, then it contains a word of length less than $|Q|$;

(iii) if $\mathcal{L}(D)$ contains all words in Σ^* of length less than $|Q|$, then $\mathcal{L}(D) = \Sigma^*$.

12B Mathematical Biology

The Fitzhugh–Nagumo model is given by

$$\begin{aligned} \dot{u} &= c\left(v + u - \tfrac{1}{3}u^3 + z(t)\right) \\ \dot{v} &= -\frac{1}{c}\,(u - a + b\,v)\,, \end{aligned}$$

where $(1 - \frac{2}{3}b) < a < 1$, $0 < b \leqslant 1$ and $c \gg 1$.

For $z(t) = 0$, by considering the nullclines in the (u, v)-plane, show that there is a unique equilibrium. Sketch the phase diagram.

At $t = 0$ the system is at the equilibrium, and $z(t)$ is then 'switched on' to be $z(t) = -V_0$ for $t > 0$, where V_0 is a constant. Describe carefully how suitable choices of $V_0 > 0$ can represent a system analogous to (i) a neuron firing once, and (ii) a neuron firing repeatedly. Illustrate your answer with phase diagrams and also plots of v against t for each case. Find the threshold for V_0 that separates these cases. Comment briefly from a biological perspective on the behaviour of the system when $a = 1 - \frac{2}{3}b + \epsilon b$ and $0 < \epsilon \ll 1$.

13C Cosmology

The early universe is described by equations (with units such that $c = 8\pi G = \hbar = 1$)

$$3H^2 = \rho\,, \qquad \dot{\rho} + 3H(\rho + p) = 0\,, \tag{1}$$

where $H = \dot{a}/a$. The universe contains only a self-interacting scalar field ϕ with interaction potential $V(\phi)$ so that the density and pressure are given by

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \\ p &= \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \end{aligned}$$

Show that

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\,. \tag{2}$$

Explain the slow-roll approximation and apply it to equations (1) and (2) to show that it leads to

$$\sqrt{3}\int \frac{\sqrt{V}}{V'}d\phi = -t + \text{const.}$$

If $V(\phi) = \frac{1}{4}\lambda\phi^4$ with λ a positive constant and $\phi(0) = \phi_0$, show that

$$\phi(t) = \phi_0 \exp\left[-\sqrt{\frac{4\lambda}{3}}\,t\right]$$

and that, for small t, the scale factor $a(t)$ expands to leading order in t as

$$a(t) \propto \exp\left[\sqrt{\frac{\lambda}{12}}\,\phi_0^2\,t\right]\,.$$

Comment on the relevance of this result for inflationary cosmology.

14F Logic and Set Theory

State the Completeness Theorem for the first-order predicate calculus, and deduce the Compactness Theorem.

Let $\mathbb{T}$ be a first-order theory over a signature Σ whose axioms all have the form $(\forall \vec{x})\phi$ where $\vec{x}$ is a (possibly empty) string of variables and ϕ is quantifier-free. Show that every substructure of a $\mathbb{T}$-model is a $\mathbb{T}$-model, and deduce that if $\mathbb{T}$ is consistent then it has a model in which every element is the interpretation of a closed term of $\mathcal{L}(\Sigma)$. [You may assume the result that if B is a substructure of A and ϕ is a quantifier-free formula with n free variables, then $[\![\phi]\!]_B = [\![\phi]\!]_A \cap B^n$.]

Now suppose $\mathbb{T} \vdash (\exists x)\psi$ where ψ is a quantifier-free formula with one free variable x. Show that there is a finite list $(t_1, t_2, \ldots, t_n)$ of closed terms of $\mathcal{L}(\Sigma)$ such that

$$\mathbb{T} \vdash (\psi[t_1/x] \vee \psi[t_2/x] \vee \cdots \vee \psi[t_n/x]).$$

15G Graph Theory

Define the *chromatic polynomial* $p_G(t)$ of a graph G. Show that if G has n vertices and m edges then

$$p_G(t) \;=\; a_n t^n - a_{n-1}t^{n-1} + a_{n-2}t^{n-2} - \ldots + (-1)^n a_0$$

where $a_n = 1$, $a_{n-1} = m$ and $a_i \geqslant 0$ for all i. [You may assume the deletion-contraction relation, provided that you state it clearly.]

Show that for every graph G (with $n > 0$) we have $a_0 = 0$. Show also that $a_1 = 0$ if and only if G is disconnected.

Explain why $t^4 - 2t^3 + 3t^2 - t$ is not the chromatic polynomial of any graph.

16H Galois Theory

(a) Let L be the 13th cyclotomic extension of $\mathbb{Q}$, and let μ be a 13th primitive root of unity. What is the minimal polynomial of μ over $\mathbb{Q}$? What is the Galois group $\mathrm{Gal}(L/\mathbb{Q})$? Put $\lambda = \mu + \frac{1}{\mu}$. Show that $\mathbb{Q} \subseteq \mathbb{Q}(\lambda)$ is a Galois extension and find $\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$.

(b) Define what is meant by a *Kummer extension*. Let K be a field of characteristic zero and let L be the nth cyclotomic extension of K. Show that there is a sequence of Kummer extensions $K = F_1 \subseteq F_2 \subseteq \cdots \subseteq F_r$ such that L is contained in F_r.

17I Representation Theory

(a) Let the finite group G act on a finite set X and let π be the permutation character. If G is 2-transitive on X, show that $\pi = 1_G + \chi$, where χ is an irreducible character of G.

(b) Let $n \geqslant 4$, and let G be the symmetric group S_n acting naturally on the set $X = \{1, \ldots, n\}$. For any integer $r \leqslant n/2$, write X_r for the set of all r-element subsets of X, and let π_r be the permutation character of the action of G on X_r. Compute the degree of π_r. If $0 \leqslant \ell \leqslant k \leqslant n/2$, compute the character inner product $\langle \pi_k, \pi_\ell \rangle$.

Let $m = n/2$ if n is even, and $m = (n-1)/2$ if n is odd. Deduce that S_n has distinct irreducible characters $\chi^{(n)} = 1_G$, $\chi^{(n-1,1)}, \chi^{(n-2,2)}, \ldots, \chi^{(n-m,m)}$ such that for all $r \leqslant m$,

$$\pi_r = \chi^{(n)} + \chi^{(n-1,1)} + \chi^{(n-2,2)} + \cdots + \chi^{(n-r,r)}.$$

(c) Let Ω be the set of all ordered pairs (i,j) with $i,j \in \{1,2,\ldots,n\}$ and $i \neq j$. Let S_n act on Ω in the obvious way. Write $\pi^{(n-2,1,1)}$ for the permutation character of S_n in this action. By considering inner products, or otherwise, prove that

$$\pi^{(n-2,1,1)} = 1 + 2\chi^{(n-1,1)} + \chi^{(n-2,2)} + \psi,$$

where ψ is an irreducible character. Calculate the degree of ψ, and calculate its value on the elements $(1\ 2)$ and $(1\ 2\ 3)$ of S_n.

18G Algebraic Topology

Construct a space X as follows. Let Z_1, Z_2, Z_3 each be homeomorphic to the standard 2-sphere $S^2 \subseteq \mathbb{R}^3$. For each i, let $x_i \in Z_i$ be the North pole $(1,0,0)$ and let $y_i \in Z_i$ be the South pole $(-1,0,0)$. Then

$$X = (Z_1 \sqcup Z_2 \sqcup Z_3)/\sim$$

where $x_{i+1} \sim y_i$ for each i (and indices are taken modulo 3).

(a) Describe the universal cover of X.

(b) Compute the fundamental group of X (giving your answer as a well-known group).

(c) Show that X is not homotopy equivalent to the circle S^1.

19I Linear Analysis

(a) Define *Banach spaces* and *Euclidean spaces* over $\mathbb{R}$. [You may assume the definitions of vector spaces and inner products.]

(b) Let X be the space of sequences of real numbers with finitely many non-zero entries. Does there exist a norm $\|\cdot\|$ on X such that $(X, \|\cdot\|)$ is a Banach space? Does there exist a norm such that $(X, \|\cdot\|)$ is Euclidean? Justify your answers.

(c) Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$ satisfying the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$. Show that $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$ is an inner product on X. [*You may use without proof the fact that the vector space operations + and $\cdot$ are continuous with respect to $\|\cdot\|$. To verify the identity $\langle a+b, c\rangle = \langle a, c\rangle + \langle b, c\rangle$, you may find it helpful to consider the parallelogram law for the pairs $(a+c, b)$, $(b+c, a)$, $(a-c, b)$ and $(b-c, a)$.*]

(d) Let $(X, \|\cdot\|_X)$ be an incomplete normed vector space over $\mathbb{R}$ which is not a Euclidean space, and let $(X^*, \|\cdot\|_{X^*})$ be its dual space with the dual norm. Is $(X^*, \|\cdot\|_{X^*})$ a Banach space? Is it a Euclidean space? Justify your answers.

20H Riemann Surfaces

Let f be a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$. Let $P \subset \mathbb{C}$ be a fundamental parallelogram and let the degree of f be n. Let $a_1, \ldots, a_n$ denote the zeros of f in P, and let $b_1, \ldots, b_n$ denote the poles (both with possible repeats). By considering the integral (if required, also slightly perturbing P)

$$\frac{1}{2\pi i}\int_{\partial P} z\frac{f'(z)}{f(z)}dz,$$

show that

$$\sum_{j=1}^{n} a_j \; - \; \sum_{j=1}^{n} b_j \; \in \Lambda.$$

Let $\wp(z)$ denote the Weierstrass $\wp$-function with respect to Λ. For $v, w \notin \Lambda$ with $\wp(v) \neq \wp(w)$ we set

$$f(z) = \det\begin{pmatrix} 1 & 1 & 1 \\ \wp(z) & \wp(v) & \wp(w) \\ \wp'(z) & \wp'(v) & \wp'(w) \end{pmatrix},$$

an elliptic function with periods Λ. Suppose $z \notin \Lambda$, $z - v \notin \Lambda$ and $z - w \notin \Lambda$. Prove that $f(z) = 0$ if and only if $z + v + w \in \Lambda$. [You may use standard properties of the Weierstrass $\wp$-function provided they are clearly stated.]

21H Algebraic Geometry

(a) Let X be an affine variety. Define the *tangent space* of X at a point P. Say what it means for the variety to be *singular* at P.

Define the *dimension* of X in terms of (i) the tangent spaces of X, and (ii) Krull dimension.

(b) Consider the ideal I generated by the set $\{y, y^2 - x^3 + xy^3\} \subseteq k[x, y]$. What is $Z(I) \subseteq \mathbb{A}^2$?

Using the generators of the ideal, calculate the tangent space of a point in $Z(I)$. What has gone wrong? [A complete argument is not necessary.]

(c) Calculate the dimension of the tangent space at each point $p \in X$ for $X = Z(x - y^2, x - zw) \subseteq \mathbb{A}^4$, and determine the location of the singularities of X.

22G Differential Geometry

Explain what it means for an embedded surface S in $\mathbf{R}^3$ to be *minimal.* What is meant by an *isothermal* parametrization $\phi : U \to V \subset \mathbf{R}^3$ of an embedded surface $V \subset \mathbf{R}^3$? Prove that if ϕ is isothermal then $\phi(U)$ is minimal if and only if the components of ϕ are harmonic functions on U. [You may assume the formula for the mean curvature of a parametrized embedded surface,

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)},$$

where E, F, G (respectively e, f, g) are the coefficients of the first (respectively second) fundamental forms.]

Let S be an embedded connected minimal surface in $\mathbf{R}^3$ which is closed as a subset of $\mathbf{R}^3$, and let $\Pi \subset \mathbf{R}^3$ be a plane which is disjoint from S. Assuming that local isothermal parametrizations always exist, show that if the Euclidean distance between S and Π is attained at some point $P \in S$, i.e. $d(P, \Pi) = \inf_{Q \in S} d(Q, \Pi)$, then S is a plane parallel to Π.

23J Probability and Measure

(a) Define the *Borel σ-algebra* $\mathcal{B}$ and the *Borel functions*.

(b) Give an example with proof of a set in $[0,1]$ which is not Lebesgue measurable.

(c) The Cantor set $\mathcal{C}$ is given by

$$\mathcal{C} = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{3^k} : (a_k) \text{ is a sequence with } a_k \in \{0,2\} \text{ for all } k \right\}.$$

(i) Explain why $\mathcal{C}$ is Lebesgue measurable.

(ii) Compute the Lebesgue measure of $\mathcal{C}$.

(iii) Is every subset of $\mathcal{C}$ Lebesgue measurable?

(iv) Let $f \colon [0,1] \to \mathcal{C}$ be the function given by

$$f(x) = \sum_{k=1}^{\infty} \frac{2a_k}{3^k} \quad \text{where} \quad a_k = \lfloor 2^k x \rfloor - 2\lfloor 2^{k-1} x \rfloor.$$

Explain why f is a Borel function.

(v) Using the previous parts, prove the existence of a Lebesgue measurable set which is not Borel.

24J Applied Probability

(a) State the thinning and superposition properties of a Poisson process on $\mathbb{R}_+$. Prove the superposition property.

(b) A bi-infinite Poisson process $(N_t : t \in \mathbb{R})$ with $N_0 = 0$ is a process with independent and stationary increments over $\mathbb{R}$. Moreover, for all $-\infty < s \leqslant t < \infty$, the increment $N_t - N_s$ has the Poisson distribution with parameter $\lambda(t-s)$. Prove that such a process exists.

(c) Let N be a bi-infinite Poisson process on $\mathbb{R}$ of intensity λ. We identify it with the set of points (S_n) of discontinuity of N, i.e., $N[s,t] = \sum_n \mathbf{1}(S_n \in [s,t])$. Show that if we shift all the points of N by the same constant c, then the resulting process is also a Poisson process of intensity λ.

Now suppose we shift every point of N by $+1$ or -1 with equal probability. Show that the final collection of points is still a Poisson process of intensity λ. [You may assume the thinning and superposition properties for the bi-infinite Poisson process.]

25J Principles of Statistics

Let $X_1, \ldots, X_n$ be i.i.d. random variables from a $N(\theta, 1)$ distribution, $\theta \in \mathbb{R}$, and consider a Bayesian model $\theta \sim N(0, v^2)$ for the unknown parameter, where $v > 0$ is a fixed constant.

(a) Derive the posterior distribution $\Pi(\cdot \mid X_1, \ldots, X_n)$ of $\theta \mid X_1, \ldots, X_n$.

(b) Construct a credible set $C_n \subset \mathbb{R}$ such that

(i) $\Pi(C_n | X_1, \ldots, X_n) = 0.95$ for every $n \in \mathbb{N}$, and

(ii) for any $\theta_0 \in \mathbb{R}$,

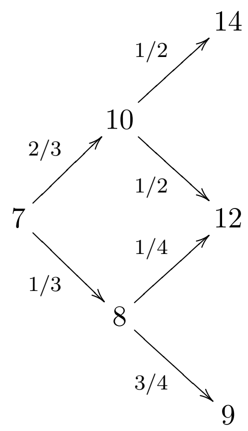
$$P_{\theta_0}^{\mathbb{N}}(\theta_0 \in C_n) \to 0.95 \quad \text{as } n \to \infty,$$

where $P_\theta^{\mathbb{N}}$ denotes the distribution of the infinite sequence $X_1, X_2, \ldots$ when drawn independently from a fixed $N(\theta, 1)$ distribution.

[*You may use the central limit theorem.*]

26K Stochastic Financial Models

Consider the following two-period market model. There is a risk-free asset which pays interest at rate $r = 1/4$. There is also a risky stock with prices $(S_t)_{t\in\{0,1,2\}}$ given by



The above diagram should be read as

$$\mathbb{P}(S_1 = 10 \mid S_0 = 7) = 2/3, \quad \mathbb{P}(S_2 = 14 \mid S_1 = 10) = 1/2$$

and so forth.

(a) Find the risk-neutral probabilities.

(b) Consider a European put option with strike $K = 10$ expiring at time $T = 2$. What is the initial no-arbitrage price of the option? How many shares should be held in each period to replicate the payout?

(c) Now consider an American put option with the same strike and expiration date. Find the optimal exercise policy, assuming immediate exercise is not allowed. Would your answer change if you were allowed to exercise the option at time 0?

27K Optimization and Control

Consider the system in scalar variables, for $t = 1, 2, \ldots, h$:

$$\begin{aligned} x_t &= x_{t-1} + u_{t-1}, \\ y_t &= x_{t-1} + \eta_t, \\ \hat{x}_0 &= x_0 + \eta_0, \end{aligned}$$

where $\hat{x}_0$ is given, y_t, u_t are observed at t, but $x_0, x_1, \ldots$ and $\eta_0, \eta_1, \ldots$ are unobservable, and $\eta_0, \eta_1, \ldots$ are independent random variables with mean 0 and variance v. Define $\hat{x}_{t-1}$ to be the estimator of x_{t-1} with minimum variance amongst all estimators that are unbiased and linear functions of $W_{t-1} = (\hat{x}_0, y_1, \ldots, y_{t-1}, u_0, \ldots, u_{t-2})$. Suppose $\hat{x}_{t-1} = a^T W_{t-1}$ and its variance is V_{t-1}. After observation at t of (y_t, u_{t-1}), a new unbiased estimator of x_{t-1}, linear in W_t, is expressed

$$x^*_{t-1} = (1 - H)b^T W_{t-1} + H y_t.$$

Find b and H to minimize the variance of x^*_{t-1}. Hence find $\hat{x}_t$ in terms of $\hat{x}_{t-1}$, y_t, u_{t-1}, V_{t-1} and v. Calculate V_h.

Suppose $\eta_0, \eta_1, \ldots$ are Gaussian and thus $\hat{x}_t = E[x_t \mid W_t]$. Consider minimizing $E[x_h^2 + \sum_{t=0}^{h-1} u_t^2]$, under the constraint that the control u_t can only depend on W_t. Show that the value function of dynamic programming for this problem can be expressed

$$F(W_t) = \Pi_t \hat{x}_t^2 + \cdots$$

where $F(W_h) = \hat{x}_h^2 + V_h$ and $+\cdots$ is independent of W_t and linear in v.

28C Asymptotic Methods

Consider the integral

$$I(x) = \int_0^1 \frac{1}{\sqrt{t(1-t)}} \exp[ixf(t)]\, dt$$

for real $x > 0$, where $f(t) = t^2 + t$. Find and sketch, in the complex t-plane, the paths of steepest descent through the endpoints $t = 0$ and $t = 1$ and through any saddle point(s). Obtain the leading order term in the asymptotic expansion of $I(x)$ for large positive x. What is the order of the next term in the expansion? Justify your answer.

29E Dynamical Systems

Consider the dependence of the system

$$\begin{aligned} \dot{x} &= (a - x^2)(a^2 - y)\,, \\ \dot{y} &= x - y \end{aligned}$$

on the parameter a. Find the fixed points and plot their location in the (a, x)-plane. Hence, or otherwise, deduce that there are bifurcations at $a = 0$ and $a = 1$.

Investigate the bifurcation at $a = 1$ by making the substitutions $u = x - 1$, $v = y - 1$ and $\mu = a - 1$. Find the extended centre manifold in the form $v(u, \mu)$ correct to second order. Find the evolution equation on the extended centre manifold to second order, and determine the stability of its fixed points.

Use a plot to show which branches of fixed points in the (a, x)-plane are stable and which are unstable, and state, without calculation, the type of bifurcation at $a = 0$. Hence sketch the structure of the (x, y) phase plane very close to the origin for $|a| \ll 1$ in the cases (i) $a < 0$ and (ii) $a > 0$.

30D Integrable Systems

What is meant by an *auto-Bäcklund* transformation?

The sine-Gordon equation in light-cone coordinates is

$$\frac{\partial^2 \varphi}{\partial \xi \partial \tau} = \sin \varphi, \tag{1}$$

where $\xi = \frac{1}{2}(x - t)$, $\tau = \frac{1}{2}(x + t)$ and φ is to be understood modulo 2π. Show that the pair of equations

$$\partial_\xi (\varphi_1 - \varphi_0) = 2\epsilon \sin\left(\frac{\varphi_1 + \varphi_0}{2}\right), \quad \partial_\tau (\varphi_1 + \varphi_0) = \frac{2}{\epsilon} \sin\left(\frac{\varphi_1 - \varphi_0}{2}\right) \tag{2}$$

constitute an auto-Bäcklund transformation for (1).

By noting that $\varphi = 0$ is a solution to (1), use the transformation (2) to derive the soliton (or 'kink') solution to the sine-Gordon equation. Show that this solution can be expressed as

$$\varphi(x,t) \;=\; 4 \arctan\left[\exp\left(\pm\frac{x - ct}{\sqrt{1-c^2}} + x_0\right)\right],$$

for appropriate constants c and x_0.

[*Hint: You may use the fact that* $\int \operatorname{cosec} x \, \mathrm{d}x = \log \tan(x/2) + \text{const.}$]

The following function is a solution to the sine-Gordon equation:

$$\varphi(x,t) \;=\; 4 \arctan\left[c\,\frac{\sinh(x/\sqrt{1-c^2})}{\cosh(ct/\sqrt{1-c^2})}\right] \quad (c > 0).$$

Verify that this represents two solitons travelling towards each other at the same speed by considering $x \pm ct = \text{constant}$ and taking an appropriate limit.

31A Principles of Quantum Mechanics

A three-dimensional oscillator has Hamiltonian

$$H = \frac{1}{2m}(\,\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2\,) \;+\; \frac{1}{2}m\omega^2(\,\alpha^2\hat{x}_1^2 + \beta^2\hat{x}_2^2 + \gamma^2\hat{x}_3^2\,)\,,$$

where the constants m, ω, α, β, γ are real and positive. Assuming a unique ground state, construct the general normalised eigenstate of H and give a formula for its energy eigenvalue. [You may quote without proof results for a one-dimensional harmonic oscillator of mass m and frequency ω that follow from writing $\hat{x} = (\hbar/2m\omega)^{1/2}(a + a^\dagger)$ and $\hat{p} = (\hbar m\omega/2)^{1/2}\, i\,(a^\dagger - a)$.]

List all states in the four lowest energy levels of H in the cases:

(i) $\alpha < \beta < \gamma < 2\alpha$;

(ii) $\alpha = \beta$ and $\gamma = \alpha + \epsilon$, where $0 < \epsilon \ll \alpha$.

Now consider H with $\alpha = \beta = \gamma = 1$ subject to a perturbation

$$\lambda m\omega^2(\,\hat{x}_1\hat{x}_2 \,+\, \hat{x}_2\hat{x}_3 \,+\, \hat{x}_3\hat{x}_1\,)\,,$$

where λ is small. Compute the changes in energies for the ground state and the states at the first excited level of the original Hamiltonian, working to the leading order at which non-zero corrections occur. [You may quote without proof results from perturbation theory.]

Explain briefly why some energy levels of the perturbed Hamiltonian will be exactly degenerate. [*Hint: Compare with* (ii) *above.*]

32A Applications of Quantum Mechanics

(a) A spinless charged particle moves in an electromagnetic field defined by vector and scalar potentials $\mathbf{A}(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$. The wavefunction $\psi(\mathbf{x},t)$ for the particle satisfies the time-dependent Schrödinger equation with Hamiltonian

$$\hat{H}_0 \;=\; \frac{1}{2m}\,(-i\hbar\boldsymbol{\nabla} + e\mathbf{A})\cdot(-i\hbar\boldsymbol{\nabla} + e\mathbf{A}) \;-\; e\phi\ .$$

Consider a gauge transformation

$$\mathbf{A} \to \tilde{\mathbf{A}} \;=\; \mathbf{A} + \boldsymbol{\nabla} f\,, \qquad \phi \to \tilde{\phi} \;=\; \phi - \frac{\partial f}{\partial t}\,, \qquad \psi \to \tilde{\psi} \;=\; \exp\Big(-\frac{ief}{\hbar}\Big)\,\psi\,,$$

for some function $f(\mathbf{x},t)$. Define *covariant derivatives* with respect to space and time, and show that $\tilde{\psi}$ satisfies the Schrödinger equation with potentials $\tilde{\mathbf{A}}$ and $\tilde{\phi}$.

(b) Suppose that in part (a) the magnetic field has the form $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = (0,0,B)$, where B is a constant, and that $\phi = 0$. Find a suitable $\mathbf{A}$ with $A_y = A_z = 0$ and determine the energy levels of the Hamiltonian $\hat{H}_0$ when the z-component of the momentum of the particle is zero. Suppose in addition that the particle is constrained to lie in a rectangular region of area $\mathcal{A}$ in the (x,y)-plane. By imposing periodic boundary conditions in the x-direction, estimate the degeneracy of each energy level. [You may use without proof results for a quantum harmonic oscillator, provided they are clearly stated.]

(c) An electron is a charged particle of spin $\frac{1}{2}$ with a two-component wavefunction $\boldsymbol{\psi}(\mathbf{x},t)$ governed by the Hamiltonian

$$\hat{H} \;=\; \hat{H}_0\,\mathbb{I}_2 + \frac{e\hbar}{2m}\,\mathbf{B}\cdot\boldsymbol{\sigma}$$

where $\mathbb{I}_2$ is the 2×2 unit matrix and $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ denotes the Pauli matrices. Find the energy levels for an electron in the constant magnetic field defined in part (b), assuming as before that the z-component of the momentum of the particle is zero.

Consider N such electrons confined to the rectangular region defined in part (b). Ignoring interactions between the electrons, show that the ground state energy of this system vanishes for N less than some integer $N_{\max}$ which you should determine. Find the ground state energy for $N = (2p+1)N_{\max}$, where p is a positive integer.

33C Statistical Physics

(a) Consider an ideal gas consisting of N identical classical particles of mass m moving freely in a volume V with Hamiltonian $H = |\mathbf{p}|^2/2m$. Show that the partition function of the gas has the form

$$Z_{\text{ideal}} = \frac{V^N}{\lambda^{3N} N!}\,,$$

and find λ as a function of the temperature T.

[You may assume that $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a}$ for $a > 0$.]

(b) A monatomic gas of interacting particles is a modification of an ideal gas where any pair of particles with separation r interact through a potential energy $U(r)$. The partition function for this gas can be written as

$$Z = Z_{\text{ideal}} \left[1 + \frac{2\pi N}{V} \int_0^{\infty} f(r)\, r^2 dr \right]^N ,$$

where $f(r) = e^{-\beta U(r)} - 1\,,\quad \beta = 1/(k_B T)$. The virial expansion of the equation of state for small densities N/V is

$$\frac{p}{k_B T} = \frac{N}{V} + B_2(T)\frac{N^2}{V^2} + \mathcal{O}\left(\frac{N^3}{V^3}\right) .$$

Using the free energy, show that

$$B_2(T) = -2\pi \int_0^{\infty} f(r)\, r^2 dr\,.$$

(c) The Lennard–Jones potential is

$$U(r) = \epsilon \left(\frac{r_0^{12}}{r^{12}} - 2\frac{r_0^6}{r^6} \right) ,$$

where ϵ and r_0 are positive constants. Find the separation σ where $U(\sigma) = 0$ and the separation r_{min} where $U(r)$ has its minimum. Sketch the graph of $U(r)$. Calculate $B_2(T)$ for this potential using the approximations

$$f(r) = e^{-\beta U(r)} - 1 \simeq \begin{cases} -1 & \text{for} \quad r < \sigma \\ -\beta U(r) & \text{for} \quad r \geqslant \sigma\,. \end{cases}$$

34E Electrodynamics

The current density in an antenna lying along the z-axis takes the form

$$\mathbf{J}(t,\mathbf{x}) = \begin{cases} \hat{\mathbf{z}}\, I_0 \,\sin\left(kd - k|z|\right) e^{-i\omega t}\delta(x)\delta(y) & |z| \leqslant d \\ \mathbf{0} & |z| > d \end{cases},$$

where I_0 is a constant and $\omega = ck$. Show that at distances $r = |\mathbf{x}|$ for which both $r \gg d$ and $r \gg kd^2/(2\pi)$, the retarded vector potential in Lorenz gauge is

$$\mathbf{A}(t,\mathbf{x}) \;\approx\; \hat{\mathbf{z}}\,\frac{\mu_0 I_0}{4\pi r}\,e^{-i\omega(t-r/c)}\int_{-d}^{d}\sin\left(kd - k|z'|\right) e^{-ikz'\cos\theta}\,dz'\,,$$

where $\cos\theta = \hat{\mathbf{r}}\cdot\hat{\mathbf{z}}$ and $\hat{\mathbf{r}} = \mathbf{x}/|\mathbf{x}|$. Evaluate the integral to show that

$$\mathbf{A}(t,\mathbf{x}) \;\approx\; \hat{\mathbf{z}}\,\frac{\mu_0 I_0}{2\pi kr}\left(\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin^2\theta}\right) e^{-i\omega(t-r/c)}\,.$$

In the far-field, where $kr \gg 1$, the electric and magnetic fields are given by

$$\mathbf{E}(t,\mathbf{x}) \;\approx\; -i\omega\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times\mathbf{A}(t,\mathbf{x})]$$
$$\mathbf{B}(t,\mathbf{x}) \;\approx\; ik\hat{\mathbf{r}}\times\mathbf{A}(t,\mathbf{x})\,.$$

By calculating the Poynting vector, show that the time-averaged power radiated per unit solid angle is

$$\frac{d\mathcal{P}}{d\Omega} = \frac{c\mu_0 I_0^2}{8\pi^2}\left(\frac{\cos(kd\cos\theta) - \cos(kd)}{\sin\theta}\right)^2.$$

[You may assume that in Lorenz gauge, the retarded potential due to a localised current distribution is

$$\mathbf{A}(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(t_{\rm ret},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\,,$$

where the retarded time $t_{\rm ret} = t - |\mathbf{x}-\mathbf{x}'|/c$.]

35D General Relativity

For a spacetime that is nearly flat, the metric g_{ab} can be expressed in the form

$$g_{ab} = \eta_{ab} + h_{ab}\,,$$

where η_{ab} is a flat metric (not necessarily diagonal) with constant components, and the components of h_{ab} and their derivatives are small. Show that

$$2R_{bd} \approx h_d{}^a{}_{,ba} + h_b{}^a{}_{,da} - h^a{}_{a,bd} - h_{bd,ac}\eta^{ac}\,,$$

where indices are raised and lowered using η_{ab}.

[You may assume that $R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}$.]

For the line element

$$ds^2 = 2du\,dv + dx^2 + dy^2 + H(u,x,y)\,du^2\,,$$

where H and its derivatives are small, show that the linearised vacuum field equations reduce to $\nabla^2 H = 0$, where ∇^2 is the two-dimensional Laplacian operator in x and y.

36B Fluid Dynamics II

A cylindrical pipe of radius a and length $L \gg a$ contains two viscous fluids arranged axisymmetrically with fluid 1 of viscosity μ_1 occupying the central region $r < \beta a$, where $0 < \beta < 1$, and fluid 2 of viscosity μ_2 occupying the surrounding annular region $\beta a < r < a$. The flow in each fluid is assumed to be steady and unidirectional, with velocities $u_1(r)\mathbf{e}_z$ and $u_2(r)\mathbf{e}_z$ respectively, with respect to cylindrical coordinates (r,θ,z) aligned with the pipe. A fixed pressure drop Δp is applied between the ends of the pipe.

Starting from the Navier–Stokes equations, derive the equations satisfied by $u_1(r)$ and $u_2(r)$, and state all the boundary conditions. Show that the pressure gradient is constant.

Solve for the velocity profile in each fluid and calculate the corresponding flow rates, Q_1 and Q_2.

Derive the relationship between β and μ_2/μ_1 that yields the same flow rate in each fluid. Comment on the behaviour of β in the limits $\mu_2/\mu_1 \gg 1$ and $\mu_2/\mu_1 \ll 1$, illustrating your comment by sketching the flow profiles.

[*Hint: In cylindrical coordinates* (r,θ,z),

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\,, \qquad e_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\,. \ ]$$

37D Waves

Small disturbances in a homogeneous elastic solid with density ρ and Lamé moduli λ and μ are governed by the equation

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u}),$$

where $\mathbf{u}(\mathbf{x},t)$ is the displacement. Show that a harmonic plane-wave solution

$$\mathbf{u} = \mathrm{Re}\left[\mathbf{A}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right]$$

must satisfy

$$\omega^2\mathbf{A} = c_P^2\,\mathbf{k}\,(\mathbf{k}\cdot\mathbf{A}) - c_S^2\,\mathbf{k}\times(\mathbf{k}\times\mathbf{A})\ ,$$

where the wavespeeds c_P and c_S are to be identified. Describe mathematically how such plane-wave solutions can be classified into longitudinal P-waves and transverse SV- and SH-waves (taking the y-direction as the vertical direction).

The half-space $y < 0$ is filled with the elastic solid described above, while the slab $0 < y < h$ is filled with a homogeneous elastic solid with Lamé moduli $\overline{\lambda}$ and $\overline{\mu}$, and wavespeeds $\overline{c}_P$ and $\overline{c}_S$. There is a rigid boundary at $y = h$. A harmonic plane SH-wave propagates from $y < 0$ towards the interface $y = 0$, with displacement

$$\mathrm{Re}\left[Ae^{i(\ell x + my - \omega t)}\right](0,0,1)\,. \qquad (*)$$

How are ℓ, m and ω related? The total displacement in $y < 0$ is the sum of $(*)$ and that of the reflected SH-wave,

$$\mathrm{Re}\left[RAe^{i(\ell x - my - \omega t)}\right](0,0,1)\ .$$

Write down the form of the displacement in $0 < y < h$, and determine the (complex) reflection coefficient R. Verify that $|R| = 1$ regardless of the parameter values, and explain this physically.

38B Numerical Analysis

(a) Define the Jacobi and Gauss–Seidel iteration schemes for solving a linear system of the form $A\mathbf{u} = \mathbf{b}$, where $\mathbf{u}, \mathbf{b} \in \mathbb{R}^M$ and $A \in \mathbb{R}^{M\times M}$, giving formulae for the corresponding iteration matrices H_J and H_{GS} in terms of the usual decomposition $A = L_0 + D + U_0$.

Show that both iteration schemes converge when A results from discretization of Poisson's equation on a square with the five-point formula, that is when

$$A = \begin{bmatrix} S & I & & & \\ I & S & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & S & I \\ & & & I & S \end{bmatrix}, \qquad S = \begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{bmatrix} \in \mathbb{R}^{m\times m} \qquad (*)$$

and $M = m^2$. [You may state the Householder–John theorem without proof.]

(b) For the matrix A given in $(*)$:

(i) Calculate the eigenvalues of H_J and deduce its spectral radius $\rho(H_J)$.

(ii) Show that each eigenvector $\mathbf{q}$ of H_{GS} is related to an eigenvector $\mathbf{p}$ of H_J by a transformation of the form $q_{i,j} = \alpha^{i+j} p_{i,j}$ for a suitable value of α.

(iii) Deduce that $\rho(H_{GS}) = \rho^2(H_J)$. What is the significance of this result for the two iteration schemes?

[*Hint: You may assume that the eigenvalues of the matrix A in $(*)$ are*

$$\lambda_{k,\ell} = -4\left(\sin^2\frac{x}{2} + \sin^2\frac{y}{2}\right), \quad \textit{where } x = \frac{k\pi h}{m+1}, \ y = \frac{\ell\pi h}{m+1}, \quad k,\ell = 1,\ldots,m,$$

with corresponding eigenvectors $\mathbf{v} = (v_{i,j})$, $\quad v_{i,j} = \sin ix \sin jy$, $\quad i,j = 1,\ldots,m$.]

END OF PAPER