UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Tuesday, 31 May, 2016 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A****,* ***B****,* ***C****, …,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Define the *Legendre symbol* and the *Jacobi symbol*. Compute the Jacobi symbols $\left(\frac{202}{11189}\right)$ and $\left(\frac{974}{1001}\right)$, stating clearly any properties of these symbols that you use.

2H Topics in Analysis

Define what it means for a subset E of $\mathbb{R}^n$ to be *convex*. Which of the following statements about a convex set E in $\mathbb{R}^n$ (with the usual norm) are always true, and which are sometimes false? Give proofs or counterexamples as appropriate.

(i) The closure of E is convex.

(ii) The interior of E is convex.

(iii) If $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then $\alpha(E)$ is convex.

(iv) If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then $f(E)$ is convex.

3G Coding and Cryptography

Show that the binary channel with channel matrix

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

has capacity $\log 5 - 2$.

4F Automata and Formal Languages

(a) Which of the following are regular languages? Justify your answers.

(i) $\{w \in \{a,b\}^* \mid w \text{ is a nonempty string of alternating } a\text{'s and } b\text{'s}\}$.

(ii) $\{wabw \mid w \in \{a,b\}^*\}$.

(b) Write down a nondeterministic finite-state automaton with ϵ-transitions which accepts the language given by the regular expression $(\mathbf{a}+\mathbf{b})^*(\mathbf{bb}+\mathbf{a})\mathbf{b}$. Describe in words what this language is.

(c) Is the following language regular? Justify your answer.

$$\{w \in \{a,b\}^* \mid w \text{ does not end in } ab \text{ or } bbb\}.$$

5K Statistical Modelling

Define an *exponential dispersion family.* Prove that the range of the natural parameter, Θ, is an open interval. Derive the mean and variance as a function of the log normalizing constant.

[*Hint: Use the convexity of* e^x*, i.e.* $e^{px+(1-p)y} \leqslant pe^x + (1-p)e^y$ *for all* $p \in [0,1]$.]

6B Mathematical Biology

(a) The populations of two competing species satisfy

$$\begin{aligned}\frac{dN_1}{dt} &= N_1[\,b_1 - \lambda(N_1+N_2)\,]\,,\\ \frac{dN_2}{dt} &= N_2[\,b_2 - \lambda(N_1+N_2)\,]\,,\end{aligned}$$

where $b_1 > b_2 > 0$ and $\lambda > 0$. Sketch the phase diagram (limiting attention to $N_1, N_2 \geqslant 0$).

The relative abundance of species 1 is defined by $U = N_1/(N_1+N_2)$. Show that

$$\frac{dU}{dt} = AU(1-U)\,,$$

where A is a constant that should be determined.

(b) Consider the spatial system

$$\frac{\partial u}{\partial t} = u(1-u) \,+\, D\frac{\partial^2 u}{\partial x^2}\,,$$

and consider a travelling-wave solution of the form $u(x,t) = f(x-ct)$ representing one species ($u=1$) invading territory previously occupied by another species ($u=0$). By linearising near the front of the invasion, show that the wave speed is given by $c = 2\sqrt{D}$.

[You may assume that the solution to the full nonlinear system will settle to the slowest possible linear wave speed.]

7A Further Complex Methods

The Euler product formula for the Gamma function is

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\ldots(z+n)}\,.$$

Use this to show that

$$\frac{\Gamma(2z)}{2^{2z}\,\Gamma(z)\,\Gamma(z+\frac{1}{2})} \;=\; c\,,$$

where c is a constant, independent of z. Find the value of c.

8E Classical Dynamics

Consider the Lagrangian

$$L = A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + B(\dot{\psi} + \dot{\phi}\cos\theta)^2 - C(\cos\theta)^k ,$$

where A, B, C are positive constants and k is a positive integer. Find three conserved quantities and show that $u = \cos\theta$ satisfies

$$\dot{u}^2 = f(u) ,$$

where $f(u)$ is a polynomial of degree $k{+}2$ which should be determined.

9C Cosmology

A spherical cloud of mass M has radius $r(t)$ and initial radius $r(0) = R$. It contains material with uniform mass density $\rho(t)$, and zero pressure. Ignoring the cosmological constant, show that if it is initially at rest at $t = 0$ and the subsequent gravitational collapse is governed by Newton's law $\ddot{r} = -GM/r^2$, then

$$\dot{r}^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right) .$$

Suppose r is given parametrically by

$$r = R\cos^2\theta ,$$

where $\theta = 0$ at $t = 0$. Derive a relation between θ and t and hence show that the cloud collapses to radius $r = 0$ at

$$t = \sqrt{\frac{3\pi}{32G\rho_0}} ,$$

where ρ_0 is the initial mass density of the cloud.

SECTION II

10H Topics in Analysis

Prove Bernstein's theorem, which states that if $f : [0,1] \to \mathbb{R}$ is continuous and

$$f_m(t) = \sum_{r=0}^{m} \binom{m}{r} f(r/m) t^r (1-t)^{m-r}$$

then $f_m(t) \to f(t)$ uniformly on $[0,1]$. [Theorems from probability theory may be used without proof provided they are clearly stated.]

Deduce Weierstrass's theorem on polynomial approximation for any closed interval.

Proving any results on Chebyshev polynomials that you need, show that, if $g : [0,\pi] \to \mathbb{R}$ is continuous and $\epsilon > 0$, then we can find an $N \geqslant 0$ and $a_j \in \mathbb{R}$, for $0 \leqslant j \leqslant N$, such that

$$\left| g(t) - \sum_{j=0}^{N} a_j \cos jt \right| \leqslant \epsilon$$

for all $t \in [0,\pi]$. Deduce that $\int_0^\pi g(t) \cos nt\, dt \to 0$ as $n \to \infty$.

11G Coding and Cryptography

Define a *BCH code* of length n, where n is odd, over the field of 2 elements with design distance δ. Show that the minimum weight of such a code is at least δ. [Results about the Vandermonde determinant may be quoted without proof, provided they are stated clearly.]

Let $\omega \in \mathbb{F}_{16}$ be a root of $X^4 + X + 1$. Let C be the BCH code of length 15 with defining set $\{\omega, \omega^2, \omega^3, \omega^4\}$. Find the generator polynomial of C and the rank of C. Determine the error positions of the following received words:

(i) $r(X) = 1 + X^6 + X^7 + X^8$,

(ii) $r(X) = 1 + X + X^4 + X^5 + X^6 + X^9$.

12A Further Complex Methods

The Hurwitz zeta function $\zeta_{\rm H}(s,q)$ is defined for $\mathrm{Re}(q)>0$ by

$$\zeta_{\rm H}(s,q) \;=\; \sum_{n=0}^{\infty} \frac{1}{(q+n)^s}\,.$$

State without proof the complex values of s for which this series converges.

Consider the integral

$$I(s,q) \;=\; \frac{\Gamma(1-s)}{2\pi i}\int_{\mathcal{C}} dz\, \frac{z^{s-1}\,e^{\,qz}}{1-e^{\,z}}$$

where $\mathcal{C}$ is the Hankel contour. Show that $I(s,q)$ provides an analytic continuation of the Hurwitz zeta function for all $s\neq 1$. Include in your account a careful discussion of removable singularities. [*Hint:* $\Gamma(s)\,\Gamma(1-s)=\pi/\sin(\pi s)$.]

Show that $I(s,q)$ has a simple pole at $s=1$ and find its residue.

13E Classical Dynamics

Define what it means for the transformation $\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ given by

$$(q_i,p_i)\;\mapsto\;\big(\,Q_i(q_j,p_j),\;P_i(q_j,p_j)\,\big)\,,\qquad i,j=1,\ldots,n$$

to be *canonical*. Show that a transformation is canonical if and only if

$$\{Q_i,Q_j\}=0\,,\quad \{P_i,P_j\}=0\,,\quad \{Q_i,P_j\}=\delta_{ij}\,.$$

Show that the transformation $\mathbb{R}^2\to\mathbb{R}^2$ given by

$$Q=q\cos\epsilon-p\sin\epsilon\,,\qquad P=q\sin\epsilon+p\cos\epsilon$$

is canonical for any real constant ϵ. Find the corresponding generating function.

14F Logic and Set Theory

Define the *von Neumann hierarchy* of sets V_α, and show that each V_α is a transitive set. Explain what is meant by saying that a binary relation on a set is *well-founded* and *extensional*. State Mostowski's Theorem.

Let r be the binary relation on ω defined by: $\langle m,n\rangle\in r$ if and only if 2^m appears in the base-2 expansion of n (i.e., the unique expression for n as a sum of distinct powers of 2). Show that r is well-founded and extensional. To which transitive set is (ω,r) isomorphic? Justify your answer.

15G Graph Theory

Define the *Turán graph* $T_r(n)$, where r and n are positive integers with $n \geqslant r$. For which r and n is $T_r(n)$ regular? For which r and n does $T_r(n)$ contain $T_4(8)$ as a subgraph?

State and prove Turán's theorem.

Let $x_1, \ldots, x_n$ be unit vectors in the plane. Prove that the number of pairs $i < j$ for which $x_i + x_j$ has length less than 1 is at most $\lfloor n^2/4 \rfloor$.

16H Galois Theory

(a) Let $K \subseteq L$ be a finite separable field extension. Show that there exist only finitely many intermediate fields $K \subseteq F \subseteq L$.

(b) Define what is meant by a *normal* extension. Is $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{1+\sqrt{7}})$ a normal extension? Justify your answer.

(c) Prove Artin's lemma, which states: if $K \subseteq L$ is a field extension, H is a finite subgroup of $\mathrm{Aut}_K(L)$, and $F := L^H$ is the fixed field of H, then $F \subseteq L$ is a Galois extension with $\mathrm{Gal}(L/F) = H$.

17I Representation Theory

Show that the 1-dimensional (complex) characters of a finite group G form a group under pointwise multiplication. Denote this group by $\widehat{G}$. Show that if $g \in G$, the map $\chi \mapsto \chi(g)$ from $\widehat{G}$ to $\mathbb{C}$ is a character of $\widehat{G}$, hence an element of $\widehat{\widehat{G}}$. What is the kernel of the map $G \to \widehat{\widehat{G}}$?

Show that if G is abelian the map $G \to \widehat{\widehat{G}}$ is an isomorphism. Deduce, from the structure theorem for finite abelian groups, that the groups G and $\widehat{G}$ are isomorphic as abstract groups.

18F Number Fields

(a) Prove that $5 + 2\sqrt{6}$ is a fundamental unit in $\mathbb{Q}(\sqrt{6})$. [You may *not* assume the continued fraction algorithm.]

(b) Determine the ideal class group of $\mathbb{Q}(\sqrt{-55})$.

19G Algebraic Topology

(a) Let K, L be simplicial complexes, and $f : |K| \to |L|$ a continuous map. What does it mean to say that $g : K \to L$ is a *simplicial approximation* to f?

(b) Define the *barycentric subdivision* of a simplicial complex K, and state the Simplicial Approximation Theorem.

(c) Show that if g is a simplicial approximation to f then $f \simeq |g|$.

(d) Show that the natural inclusion $|K^{(1)}| \to |K|$ induces a surjective map on fundamental groups.

20I Linear Analysis

(a) Let K be a topological space and let $C_{\mathbb{R}}(K)$ denote the normed vector space of bounded continuous real-valued functions on K with the norm $\|f\|_{C_{\mathbb{R}}(K)} = \sup_{x \in K} |f(x)|$. Define the terms *uniformly bounded*, *equicontinuous* and *relatively compact* as applied to subsets $S \subset C_{\mathbb{R}}(K)$.

(b) The Arzela–Ascoli theorem [which you need not prove] states in particular that if K is compact and $S \subset C_{\mathbb{R}}(K)$ is uniformly bounded and equicontinuous, then S is relatively compact. Show by examples that each of the compactness of K, uniform boundedness of S, and equicontinuity of S are necessary conditions for this conclusion.

(c) Let L be a topological space. Assume that there exists a sequence of compact subsets K_n of L such that $K_1 \subset K_2 \subset K_3 \subset \cdots \subset L$ and $\bigcup_{n=1}^{\infty} K_n = L$. Suppose $S \subset C_{\mathbb{R}}(L)$ is uniformly bounded and equicontinuous and moreover satisfies the condition that, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $|f(x)| < \epsilon$ for every $x \in L \setminus K_n$ and for every $f \in S$. Show that S is relatively compact.

21H Riemann Surfaces

Suppose that $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is a holomorphic map of complex tori, and let π_j denote the projection map $\mathbb{C} \to \mathbb{C}/\Lambda_j$ for $j = 1, 2$. Show that there is a holomorphic map $F : \mathbb{C} \to \mathbb{C}$ such that $\pi_2 F = f \pi_1$.

Prove that $F(z) = \lambda z + \mu$ for some $\lambda, \mu \in \mathbb{C}$. Hence deduce that two complex tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are conformally equivalent if and only if the lattices are related by $\Lambda_2 = \lambda \Lambda_1$ for some $\lambda \in \mathbb{C}^*$.

22H Algebraic Geometry

In this question we work over an algebraically closed field of characteristic zero. Let $X^o = Z(x^6 + xy^5 + y^6 - 1) \subset \mathbb{A}^2$ and let $X \subset \mathbb{P}^2$ be the closure of X^o in $\mathbb{P}^2$.

(a) Show that X is a non-singular curve.

(b) Show that $\omega = dx/(5xy^4 + 6y^5)$ is a regular differential on X.

(c) Compute the divisor of ω. What is the genus of X?

23G Differential Geometry

If an embedded surface $S \subset \mathbf{R}^3$ contains a line L, show that the Gaussian curvature is non-positive at each point of L. Give an example where the Gaussian curvature is zero at each point of L.

Consider the helicoid S given as the image of $\mathbf{R}^2$ in $\mathbf{R}^3$ under the map

$$\phi(u, v) = (\sinh v \, \cos u, \, \sinh v \, \sin u, \, u).$$

What is the image of the corresponding Gauss map? Show that the Gaussian curvature at a point $\phi(u, v) \in S$ is given by $-1/\cosh^4 v$, and hence is strictly negative everywhere. Show moreover that there is a line in S passing through any point of S.

[*General results concerning the first and second fundamental forms on an oriented embedded surface $S \subset \mathbf{R}^3$ and the Gauss map may be used without proof in this question.*]

24J Probability and Measure

(a) State Jensen's inequality. Give the definition of $\| \cdot \|_{L^p}$ and the space L^p for $1 < p < \infty$. If $\|f - g\|_{L^p} = 0$, is it true that $f = g$? Justify your answer. State and prove Hölder's inequality using Jensen's inequality.

(b) Suppose that $(E, \mathcal{E}, \mu)$ is a finite measure space. Show that if $1 < q < p$ and $f \in L^p(E)$ then $f \in L^q(E)$. Give the definition of $\| \cdot \|_{L^\infty}$ and show that $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$.

(c) Suppose that $1 < q < p < \infty$. Show that if f belongs to both $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, then $f \in L^r(\mathbb{R})$ for any $r \in [q, p]$. If $f \in L^p(\mathbb{R})$, must we have $f \in L^q(\mathbb{R})$? Give a proof or a counterexample.

25J Applied Probability

(a) Define an $M/M/\infty$ *queue* and write without proof its stationary distribution. State Burke's theorem for an $M/M/\infty$ queue.

(b) Let X be an $M/M/\infty$ queue with arrival rate λ and service rate μ started from the stationary distribution. For each t, denote by $A_1(t)$ the last time before t that a customer departed the queue and $A_2(t)$ the first time after t that a customer departed the queue. If there is no arrival before time t, then we set $A_1(t) = 0$. What is the limit as $t \to \infty$ of $\mathbb{E}[A_2(t) - A_1(t)]$? Explain.

(c) Consider a system of N queues serving a finite number K of customers in the following way: at station $1 \leqslant i \leqslant N$, customers are served immediately and the service times are independent exponentially distributed with parameter μ_i; after service, each customer goes to station j with probability $p_{ij} > 0$. We assume here that the system is closed, i.e., $\sum_j p_{ij} = 1$ for all $1 \leqslant i \leqslant N$.

Let $S = \{(n_1, \ldots, n_N) : n_i \in \mathbb{N}, \sum_{i=1}^N n_i = K\}$ be the state space of the Markov chain. Write down its Q-matrix. Also write down the Q-matrix R corresponding to the position in the network of one customer (that is, when $K = 1$). Show that there is a unique distribution $(\lambda_i)_{1 \leqslant i \leqslant N}$ such that $\lambda R = 0$. Show that

$$\pi(n) = C_N \prod_{i=1}^N \frac{\lambda_i^{n_i}}{n_i!}, \quad n = (n_1, \ldots, n_N) \in S,$$

defines an invariant measure for the chain. Are the queue lengths independent at equilibrium?

26J Principles of Statistics

(a) State and prove the Cramér–Rao inequality in a parametric model $\{f(\theta) : \theta \in \Theta\}$, where $\Theta \subseteq \mathbb{R}$. [Necessary regularity conditions on the model need not be specified.]

(b) Let $X_1, \ldots, X_n$ be i.i.d. Poisson random variables with unknown parameter $EX_1 = \theta > 0$. For $\bar{X}_n = (1/n) \sum_{i=1}^n X_i$ and $S^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ define

$$T_\alpha = \alpha \bar{X}_n + (1-\alpha) S^2, \quad 0 \leqslant \alpha \leqslant 1.$$

Show that $\mathrm{Var}_\theta(T_\alpha) \geqslant \mathrm{Var}_\theta(\bar{X}_n)$ for all values of α, θ.

Now suppose $\tilde{\theta} = \tilde{\theta}(X_1, \ldots, X_n)$ is an estimator of θ with possibly nonzero bias $B(\theta) = E_\theta \tilde{\theta} - \theta$. Suppose the function B is monotone increasing on $(0, \infty)$. Prove that the mean-squared errors satisfy

$$E_\theta(\tilde{\theta}_n - \theta)^2 \geqslant E_\theta(\bar{X}_n - \theta)^2 \quad \text{for all } \theta \in \Theta.$$

27K Stochastic Financial Models

In the context of the Black–Scholes model, let S_0 be the initial price of the stock, and let σ be its volatility. Assume that the risk-free interest rate is zero and the stock pays no dividends. Let $\mathrm{EC}(S_0, K, \sigma, T)$ denote the initial price of a European call option with strike K and maturity date T.

(a) Show that the Black–Scholes formula can be written in the form

$$\mathrm{EC}(S_0, K, \sigma, T) = S_0 \Phi(d_1) - K\Phi(d_2),$$

where d_1 and d_2 depend on S_0, K, σ and T, and Φ is the standard normal distribution function.

(b) Let $\mathrm{EP}(S_0, K, \sigma, T)$ be the initial price of a put option with strike K and maturity T. Show that

$$\mathrm{EP}(S_0, K, \sigma, T) = \mathrm{EC}(S_0, K, \sigma, T) + K - S_0\,.$$

(c) Show that

$$\mathrm{EP}(S_0, K, \sigma, T) = \mathrm{EC}(K, S_0, \sigma, T)\,.$$

(d) Consider a European contingent claim with maturity T and payout

$$S_T I_{\{S_T \leqslant K\}} - K I_{\{S_T > K\}}\,.$$

Assuming $K > S_0$, show that its initial price can be written as $\mathrm{EC}(S_0, K, \hat{\sigma}, T)$ for a volatility parameter $\hat{\sigma}$ which you should express in terms of S_0, K, σ and T.

28K Optimization and Control

Consider a Markov decision problem with finite state space X, value function F and dynamic programming equation $F = \mathcal{L}F$, where

$$(\mathcal{L}\phi)(i) = \min_{a \in \{0,1\}} \Big\{ c(i,a) + \beta \textstyle\sum_{j \in X} P_{ij}(a)\phi(j) \Big\}.$$

Suppose $0 < \beta < 1$, and $|c(i,a)| \leqslant B$ for all $i \in X$, $a \in \{0,1\}$. Prove there exists a *deterministic stationary Markov policy* that is *optimal*, explaining what the italicised words mean.

Let $F_n = \mathcal{L}^n F_0$, where $F_0 = 0$, and $M_n = \max_{i \in X} |F(i) - F_n(i)|$. Prove that

$$M_n \leqslant \beta M_{n-1} \leqslant \beta^n B/(1-\beta).$$

Deduce that the value iteration algorithm converges to an optimal policy in a finite number of iterations.

29C Asymptotic Methods

What is meant by the asymptotic relation

$$f(z) \sim g(z) \quad \text{as} \quad z \to z_0\,,\ \mathrm{Arg}\,(z-z_0) \in (\theta_0, \theta_1)\,?$$

Show that

$$\sinh(z^{-1}) \sim \frac{1}{2}\exp(z^{-1}) \quad \text{as} \quad z \to 0\,,\ \mathrm{Arg}\,z \in (-\pi/2, \pi/2)\,,$$

and find the corresponding result in the sector $\mathrm{Arg}\,z \in (\pi/2, 3\pi/2)$.

What is meant by the asymptotic expansion

$$f(z) \sim \sum_{j=0}^{\infty} c_j (z-z_0)^j \quad \text{as} \quad z \to z_0\,,\ \mathrm{Arg}\,(z-z_0) \in (\theta_0, \theta_1)\,?$$

Show that the coefficients $\{c_j\}_{j=0}^{\infty}$ are determined uniquely by f. Show that if f is analytic at z_0, then its Taylor series is an asymptotic expansion for f as $z \to z_0$ (for any $\mathrm{Arg}\,(z-z_0)$).

Show that

$$u(x,t) = \int_{-\infty}^{\infty} \exp(-ik^2 t + ikx)\, f(k)\, dk$$

defines a solution of the equation $i\,\partial_t u + \partial_x^2 u = 0$ for any smooth and rapidly decreasing function f. Use the method of stationary phase to calculate the leading-order behaviour of $u(\lambda t, t)$ as $t \to +\infty$, for fixed λ.

30E Dynamical Systems

Consider the nonlinear oscillator

$$\begin{aligned} \dot{x} &= y - \mu x(\tfrac{1}{2}|x| - 1)\,, \\ \dot{y} &= -x\,. \end{aligned}$$

(a) Use the Hamiltonian for $\mu = 0$ to find a constraint on the size of the domain of stability of the origin when $\mu < 0$.

(b) Assume that given $\mu > 0$ there exists an R such that all trajectories eventually remain within the region $|\mathbf{x}| \leqslant R$. Show that there must be a limit cycle, stating carefully any result that you use. [You need not show that there is only one periodic orbit.]

(c) Use the energy-balance method to find the approximate amplitude of the limit cycle for $0 < \mu \ll 1$.

(d) Find the approximate shape of the limit cycle for $\mu \gg 1$, and calculate the leading-order approximation to its period.

31D Integrable Systems

What does it mean for $g^\epsilon : (x,u) \mapsto (\tilde{x},\tilde{u})$ to describe a *1-parameter group of transformations*? Explain how to compute the vector field

$$V = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} \qquad (*)$$

that generates such a 1-parameter group of transformations.

Suppose now $u = u(x)$. Define the nth prolongation, $\mathrm{pr}^{(n)} g^\epsilon$, of g^ϵ and the vector field which generates it. If V is defined by $(*)$ show that

$$\mathrm{pr}^{(n)} V \;=\; V \;+\; \sum_{k=1}^{n} \eta_k \, \frac{\partial}{\partial u^{(k)}}\,,$$

where $u^{(k)} = \mathrm{d}^k u/\mathrm{d}x^k$ and η_k are functions to be determined.

The curvature of the curve $u = u(x)$ in the (x,u)-plane is given by

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}\,.$$

Rotations in the (x,u)-plane are generated by the vector field

$$W = x\frac{\partial}{\partial u} - u\frac{\partial}{\partial x}\,.$$

Show that the curvature κ at a point along a plane curve is invariant under such rotations. Find two further transformations that leave κ invariant.

32A Principles of Quantum Mechanics

(a) Let $|\, j\, m\,\rangle$ be standard, normalised angular momentum eigenstates with labels specifying eigenvalues for $\mathbf{J}^2$ and J_3. Taking units in which $\hbar = 1$,

$$J_\pm |\, j\, m\,\rangle \;=\; \big\{\, (j \mp m)(j \pm m + 1)\,\big\}^{1/2}\, |\, j\, m \pm 1\,\rangle\ .$$

Check the coefficients above by computing norms of states, quoting any angular momentum commutation relations that you require.

(b) Two particles, each of spin $s > 0$, have combined spin states $|\, J\, M\,\rangle$. Find expressions for all such states with $M = 2s-1$ in terms of product states.

(c) Suppose that the particles in part (b) move about their centre of mass with a spatial wavefunction that is a spherically symmetric function of relative position. If the particles are identical, what spin states $|\, J\ 2s-1\,\rangle$ are allowed? Justify your answer.

(d) Now consider two particles of spin 1 that are not identical and are both at rest. If the 3-component of the spin of each particle is zero, what is the probability that their total, combined spin is zero?

33A Applications of Quantum Mechanics

A particle of mass m moves in three dimensions subject to a potential $V(\mathbf{r})$ localised near the origin. The wavefunction for a scattering process with incident particle of wavevector $\mathbf{k}$ is denoted $\psi(\mathbf{k}, \mathbf{r})$. With reference to the asymptotic form of ψ, define the scattering amplitude $f(\mathbf{k}, \mathbf{k}')$, where $\mathbf{k}'$ is the wavevector of the outgoing particle with $|\mathbf{k}'| = |\mathbf{k}| = k$.

By recasting the Schrödinger equation for $\psi(\mathbf{k}, \mathbf{r})$ as an integral equation, show that

$$f(\mathbf{k}, \mathbf{k}') \;=\; -\frac{m}{2\pi\hbar^2} \int d^3\mathbf{r}' \exp(-i\mathbf{k}' \cdot \mathbf{r}')\, V(\mathbf{r}')\, \psi(\mathbf{k}, \mathbf{r}')\ .$$

[You may assume that

$$\mathcal{G}(k; \mathbf{r}) \;=\; -\frac{1}{4\pi|\mathbf{r}|} \exp(\, ik|\mathbf{r}|\,)$$

is the Green's function for $\nabla^2 + k^2$ which obeys the appropriate boundary conditions for a scattering solution.]

Now suppose $V(\mathbf{r}) = \lambda\, U(\mathbf{r})$, where $\lambda \ll 1$ is a dimensionless constant. Determine the first two non-zero terms in the expansion of $f(\mathbf{k}, \mathbf{k}')$ in powers of λ, giving each term explicitly as an integral over one or more position variables $\mathbf{r}$, $\mathbf{r}'$, $\ldots$.

Evaluate the contribution to $f(\mathbf{k}, \mathbf{k}')$ of order λ in the case $U(\mathbf{r}) = \delta(\,|\mathbf{r}| - a\,)$, expressing the answer as a function of a, k and the scattering angle θ (defined so that $\mathbf{k} \cdot \mathbf{k}' = k^2 \cos\theta$).

34C Statistical Physics

(a) What is meant by the *canonical ensemble*? Consider a system in the canonical ensemble that can be in states $|n\rangle$, $n = 0, 1, 2, \dots$ with energies E_n. Write down the partition function for this system and the probability $p(n)$ that the system is in state $|n\rangle$. Derive an expression for the average energy $\langle E\rangle$ in terms of the partition function.

(b) Consider an anharmonic oscillator with energy levels

$$\hbar\omega\left[\left(n+\frac{1}{2}\right)+\delta\left(n+\frac{1}{2}\right)^2\right], \qquad n = 0, 1, 2, \dots,$$

where ω is a positive constant and $0 < \delta \ll 1$ is a small constant. Let the oscillator be in contact with a reservoir at temperature T. Show that, to linear order in δ, the partition function Z_1 for the oscillator is given by

$$Z_1 = \frac{c_1}{\sinh\frac{x}{2}}\left[1 + \delta\, c_2\, x\left(1+\frac{2}{\sinh^2\frac{x}{2}}\right)\right], \qquad x = \frac{\hbar\omega}{k_B T},$$

where c_1 and c_2 are constants to be determined. Also show that, to linear order in δ, the average energy of a system of N uncoupled oscillators of this type is given by

$$\langle E\rangle = \frac{N\hbar\omega}{2}\left\{c_3\coth\frac{x}{2} + \delta\left[c_4 + \frac{c_5}{\sinh^2\frac{x}{2}}\left(1 - x\coth\frac{x}{2}\right)\right]\right\},$$

where c_3, c_4, c_5 are constants to be determined.

35D General Relativity

The Kasner (vacuum) cosmological model is defined by the line element

$$ds^2 = -c^2 dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 \qquad \text{with} \qquad t > 0\,,$$

where p_1, p_2, p_3 are constants with $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$ and $0 < p_1 < 1$. Show that $p_2\, p_3 < 0$.

Write down four equations that determine the null geodesics of the Kasner model.

If k^a is the tangent vector to the trajectory of a photon and u^a is the four-velocity of a comoving observer (i.e., an observer at rest in the (t, x, y, z) coordinate system above), what is the physical interpretation of $k_a u^a$?

Let O be a comoving observer at the origin, $x = y = z = 0$, and let S be a comoving source of photons located on one of the spatial coordinate axes.

(i) Show that photons emitted by S and observed by O can be either red-shifted or blue-shifted, depending on the location of S.

(ii) Given any fixed time $t = T$, show that there are locations for S on each coordinate axis from which no photons reach O for $t \leqslant T$.

Now suppose that $p_1 = 1$ and $p_2 = p_3 = 0$. Does the property in (ii) still hold?

36B Fluid Dynamics II

For a two-dimensional flow in plane polar coordinates (r, θ), state the relationship between the streamfunction $\psi(r, \theta)$ and the flow components u_r and u_θ. Show that the vorticity ω is given by $\omega = -\nabla^2\psi$, and deduce that the streamfunction for a steady two-dimensional Stokes flow satisfies the biharmonic equation

$$\nabla^4\psi = 0\,.$$

A rigid stationary circular disk of radius a occupies the region $r \leqslant a$. The flow far from the disk tends to a steady straining flow $\mathbf{u}_\infty = (-Ex, Ey)$, where E is a constant. Inertial forces may be neglected. Calculate the streamfunction, $\psi_\infty(r, \theta)$, for the far-field flow.

By making an appropriate assumption about its dependence on θ, find the streamfunction ψ for the flow around the disk, and deduce the flow components, $u_r(r, \theta)$ and $u_\theta(r, \theta)$.

Calculate the tangential surface stress, $\sigma_{r\theta}$, acting on the boundary of the disk.

[*Hints: In plane polar coordinates* (r, θ),

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\,, \qquad \omega = \frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\,,$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}\,, \qquad e_{r\theta} = \frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\,.\ ]$$

37D Waves

Starting from the equations for one-dimensional unsteady flow of a perfect gas at constant entropy, show that the Riemann invariants

$$R_{\pm} = u \pm \frac{2(c - c_0)}{\gamma - 1}$$

are constant on characteristics $C_{\pm}$ given by $dx/dt = u \pm c$, where $u(x,t)$ is the speed of the gas, $c(x,t)$ is the local speed of sound, c_0 is a constant and $\gamma > 1$ is the exponent in the adiabatic equation of state for $p(\rho)$.

At time $t = 0$ the gas occupies $x > 0$ and is at rest at uniform density ρ_0, pressure p_0 and sound speed c_0. For $t > 0$, a piston initially at $x = 0$ has position $x = X(t)$, where

$$X(t) = -U_0\, t\Big(1 - \frac{t}{2t_0}\Big)$$

and U_0 and t_0 are positive constants. For the case $0 < U_0 < 2c_0/(\gamma - 1)$, sketch the piston path $x = X(t)$ and the C_+ characteristics in $x \geqslant X(t)$ in the (x,t)-plane, and find the time and place at which a shock first forms in the gas.

Do likewise for the case $U_0 > 2c_0/(\gamma - 1)$.

38B Numerical Analysis

(a) The advection equation

$$u_t = u_x, \quad 0 \leqslant x \leqslant 1,\ t \geqslant 0$$

is discretised using an equidistant grid with stepsizes $\Delta x = h$ and $\Delta t = k$. The spatial derivatives are approximated with central differences and the resulting ODEs are approximated with the trapezoidal rule. Write down the relevant difference equation for determining (u_m^{n+1}) from (u_m^n). What is the name of this scheme? What is the local truncation error?

The boundary condition is periodic, $u(0,t) = u(1,t)$. Explain briefly how to write the discretised scheme in the form $B\mathbf{u}^{n+1} = C\mathbf{u}^n$, where the matrices B and C, to be identified, have a circulant form. Using matrix analysis, find the range of $\mu = \Delta t/\Delta x$ for which the scheme is stable. [Standard results may be used without proof if quoted carefully.]

[*Hint: An* $n \times n$ circulant *matrix has the form*

$$A = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ a_1 & \dots & a_{n-1} & a_0 \end{pmatrix}.$$

All such matrices have the same set of eigenvectors $\mathbf{v}_\ell = \left(\omega^{j\ell}\right)_{j=0}^{n-1}$, $\ell = 0,1,\dots,n-1$, *where* $\omega = e^{2\pi i/n}$, *and the corresponding eigenvalues are* $\lambda_\ell = \sum_{k=0}^{n-1} a_k \omega^{k\ell}$.]

(b) Consider the advection equation on the unit square

$$u_t = au_x + bu_y, \quad 0 \leqslant x, y \leqslant 1,\ t \geqslant 0,$$

where u satisfies doubly periodic boundary conditions, $u(0,y) = u(1,y)$, $u(x,0) = u(x,1)$, and $a(x,y)$ and $b(x,y)$ are given doubly periodic functions. The system is discretised with the Crank–Nicolson scheme, with central differences for the space derivatives, using an equidistant grid with stepsizes $\Delta x = \Delta y = h$ and $\Delta t = k$. Write down the relevant difference equation, and show how to write the scheme in the form

$$\mathbf{u}^{n+1} = (I - \tfrac{1}{4}\mu A)^{-1}(I + \tfrac{1}{4}\mu A)\mathbf{u}^n, \qquad (*)$$

where the matrix A should be identified. Describe how $(*)$ can be approximated by Strang splitting, and explain the advantages of doing so.

[*Hint: Inversion of the matrix* B *in part (a) has a similar computational cost to that of a tridiagonal matrix.*]

END OF PAPER