UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday, 1 June, 2016 9:00 am to 12:00 pm

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, ..., H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Find a linear change of coordinates such that the quadratic form

$$2x^2 + 8xy - 6xz + y^2 - 4yz + 2z^2$$

takes the form

$$\alpha x^2 + \beta y^2 + \gamma z^2,$$

for real numbers α, β and γ.

2E Groups, Rings and Modules

Let R be an integral domain.

Define what is meant by the *field of fractions* F of R. [You do not need to prove the existence of F.]

Suppose that $\phi : R \to K$ is an injective ring homomorphism from R to a field K. Show that ϕ extends to an injective ring homomorphism $\Phi : F \to K$.

Give an example of R and a ring homomorphism $\psi : R \to S$ from R to a ring S such that ψ does not extend to a ring homomorphism $F \to S$.

3G Analysis II

(a) What does it mean to say that the function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at the point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$? Show from your definition that if f is differentiable at x, then f is continuous at x.

(b) Suppose that there are functions $g_j\colon \mathbb{R} \to \mathbb{R}^m$ $(1 \leqslant j \leqslant n)$ such that for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$f(x) = \sum_{j=1}^{n} g_j(x_j).$$

Show that f is differentiable at x if and only if each g_j is differentiable at x_j.

(c) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = |x|^{3/2} + |y|^{1/2}.$$

Determine at which points $(x, y) \in \mathbb{R}^2$ the function f is differentiable.

4E Metric and Topological Spaces

Consider $\mathbb{R}$ and $\mathbb{Q}$ with their usual topologies.

(a) Show that compact subsets of a Hausdorff topological space are closed. Show that compact subsets of $\mathbb{R}$ are closed and bounded.

(b) Show that there exists a complete metric space (X, d) admitting a surjective continuous map $f\colon X \to \mathbb{Q}$.

5A Methods

Use the method of characteristics to find $u(x, y)$ in the first quadrant $x \geqslant 0$, $y \geqslant 0$, where $u(x, y)$ satisfies

$$\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial y} = \cos x,$$

with boundary data $u(x, 0) = \cos x$.

6D Electromagnetism

(a) Derive the integral form of Ampère's law from the differential form of Maxwell's equations with a time-independent magnetic field, $\rho = 0$ and $\mathbf{E} = \mathbf{0}$.

(b) Consider two perfectly-conducting concentric thin cylindrical shells of infinite length with axes along the z-axis and radii a and b $(a < b)$. Current I flows in the positive z-direction in each shell. Use Ampère's law to calculate the magnetic field in the three regions: (i) $r < a$, (ii) $a < r < b$ and (iii) $r > b$, where $r = \sqrt{x^2 + y^2}$.

(c) If current I now flows in the positive z-direction in the inner shell and in the negative z-direction in the outer shell, calculate the magnetic field in the same three regions.

7C Fluid Dynamics

A steady, two-dimensional unidirectional flow of a fluid with dynamic viscosity μ is set up between two plates at $y = 0$ and $y = h$. The plate at $y = 0$ is stationary while the plate at $y = h$ moves with constant speed $U\mathbf{e}_x$. The fluid is also subject to a constant pressure gradient $-G\mathbf{e}_x$. You may assume that the fluid velocity $\mathbf{u}$ has the form $\mathbf{u} = u(y)\mathbf{e}_x$.

(a) State the equation satisfied by $u(y)$ and its boundary conditions.

(b) Calculate $u(y)$.

(c) Show that the value of U may be chosen to lead to zero viscous shear stress acting on the bottom plate and calculate the resulting flow rate.

8H Statistics

The efficacy of a new medicine was tested as follows. Fifty patients were given the medicine, and another fifty patients were given a placebo. A week later, the number of patients who got better, stayed the same, or got worse was recorded, as summarised in this table:

	medicine	placebo
better	28	22
same	4	16
worse	18	12

Conduct a Pearson chi-squared test of size 1% of the hypothesis that the medicine and the placebo have the same effect.

[*Hint: You may find the following values relevant:*

Distribution	χ_1^2	χ_2^2	χ_3^2	χ_4^2	χ_5^2	χ_6^2
99% percentile	*6.63*	*9.21*	*11.34*	*13.3*	*15.09*	*16.81.*

]

9H Optimization

Use the simplex algorithm to find the optimal solution to the linear program:

$$\text{maximise } 3x+5y \text{ subject to } \begin{array}{rcrcrcl} 8x & + & 3y & + & 10z & \leqslant & 9, \quad x,y,z \geqslant 0 \\ 5x & + & 2y & + & 4z & \leqslant & 8 \\ 2x & + & y & + & 3z & \leqslant & 2. \end{array}$$

Write down the dual problem and find its solution.

SECTION II

10F Linear Algebra

Let $M_{n,n}$ denote the vector space over a field $F = \mathbb{R}$ or $\mathbb{C}$ of $n \times n$ matrices with entries in F. Given $B \in M_{n,n}$, consider the two linear transformations $R_B, L_B : M_{n,n} \to M_{n,n}$ defined by

$$L_B(A) = BA, \qquad R_B(A) = AB.$$

(a) Show that $\det L_B = (\det B)^n$.

[For parts (b) and (c), you may assume the analogous result $\det R_B = (\det B)^n$ without proof.]

(b) Now let $F = \mathbb{C}$. For $B \in M_{n,n}$, write B^* for the conjugate transpose of B, i.e., $B^* := \overline{B}^T$. For $B \in M_{n,n}$, define the linear transformation $M_B : M_{n,n} \to M_{n,n}$ by

$$M_B(A) = BAB^*.$$

Show that $\det M_B = |\det B|^{2n}$.

(c) Again let $F = \mathbb{C}$. Let $W \subseteq M_{n,n}$ be the set of Hermitian matrices. [Note that W is not a vector space over $\mathbb{C}$ but only over $\mathbb{R}$.] For $B \in M_{n,n}$ and $A \in W$, define $T_B(A) = BAB^*$. Show that T_B is an $\mathbb{R}$-linear operator on W, and show that as such,

$$\det T_B = |\det B|^{2n}.$$

11E Groups, Rings and Modules

(a) State Sylow's theorems and give the proof of the second theorem which concerns conjugate subgroups.

(b) Show that there is no simple group of order 351.

(c) Let k be the finite field $\mathbb{Z}/(31)$ and let $GL_2(k)$ be the multiplicative group of invertible 2×2 matrices over k. Show that every Sylow 3-subgroup of $GL_2(k)$ is abelian.

12G Analysis II

(a) What is a *norm* on a real vector space?

(b) Let $L(\mathbb{R}^m, \mathbb{R}^n)$ be the space of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. Show that

$$\|A\| = \sup_{0 \neq x \in \mathbb{R}^m} \frac{\|Ax\|}{\|x\|}, \qquad A \in L(\mathbb{R}^m, \mathbb{R}^n),$$

defines a norm on $L(\mathbb{R}^m, \mathbb{R}^n)$, and that if $B \in L(\mathbb{R}^\ell, \mathbb{R}^m)$ then $\|AB\| \leqslant \|A\|\, \|B\|$.

(c) Let M_n be the space of $n \times n$ real matrices, identified with $L(\mathbb{R}^n, \mathbb{R}^n)$ in the usual way. Let $U \subset M_n$ be the subset

$$U = \{X \in M_n \mid I - X \text{ is invertible}\}\,.$$

Show that U is an open subset of M_n which contains the set $V = \{X \in M_n \mid \|X\| < 1\}$.

(d) Let $f \colon U \to M_n$ be the map $f(X) = (I - X)^{-1}$. Show carefully that the series $\sum_{k=0}^{\infty} X^k$ converges on V to $f(X)$. Hence or otherwise, show that f is twice differentiable at 0, and compute its first and second derivatives there.

13A Complex Analysis or Complex Methods

Let $a = N + 1/2$ for a positive integer N. Let C_N be the anticlockwise contour defined by the square with its four vertices at $a \pm ia$ and $-a \pm ia$. Let

$$I_N = \oint_{C_N} \frac{dz}{z^2 \sin(\pi z)}.$$

Show that $1/\sin(\pi z)$ is uniformly bounded on the contours C_N as $N \to \infty$, and hence that $I_N \to 0$ as $N \to \infty$.

Using this result, establish that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

14F Geometry

(a) Let ABC be a hyperbolic triangle, with the angle at A at least $\pi/2$. Show that the side BC has maximal length amongst the three sides of ABC.
[You may use the hyperbolic cosine formula without proof. This states that if a, b and c are the lengths of BC, AC, and AB respectively, and α, β and γ are the angles of the triangle at A, B and C respectively, then

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha.]$$

(b) Given points z_1, z_2 in the hyperbolic plane, let w be any point on the hyperbolic line segment joining z_1 to z_2, and let w' be any point not on the hyperbolic line passing through z_1, z_2, w. Show that

$$\rho(w', w) \leqslant \max\{\rho(w', z_1), \rho(w', z_2)\},$$

where ρ denotes hyperbolic distance.

(c) The diameter of a hyperbolic triangle Δ is defined to be

$$\sup\{\rho(P, Q) \mid P, Q \in \Delta\}.$$

Show that the diameter of Δ is equal to the length of its longest side.

15C Variational Principles

A flexible wire filament is described by the curve $(x, y(x), z(x))$ in cartesian coordinates for $0 \leqslant x \leqslant L$. The filament is assumed to be almost straight and thus we assume $|y'| \ll 1$ and $|z'| \ll 1$ everywhere.

(a) Show that the total length of the filament is approximately $L + \Delta$ where

$$\Delta = \frac{1}{2} \int_0^L \left[(y')^2 + (z')^2 \right] dx.$$

(b) Under a uniform external axial force, $F > 0$, the filament adopts the shape which minimises the total energy, $\mathcal{E} = E_B - F\Delta$, where E_B is the bending energy given by

$$E_B[y, z] = \frac{1}{2} \int_0^L \left[A(x)(y'')^2 + B(x)(z'')^2 \right] dx,$$

and where $A(x)$ and $B(x)$ are x-dependent bending rigidities (both known and strictly positive). The filament satisfies the boundary conditions

$$y(0) = y'(0) = z(0) = z'(0) = 0, \quad y(L) = y'(L) = z(L) = z'(L) = 0.$$

Derive the Euler-Lagrange equations for $y(x)$ and $z(x)$.

(c) In the case where $A = B = 1$ and $L = 1$, show that below a critical force, F_c, which should be determined, the only energy-minimising solution for the filament is straight ($y = z = 0$), but that a new nonzero solution is admissible at $F = F_c$.

16A Methods

Consider a bar of length π with free ends, subject to longitudinal vibrations. You may assume that the longitudinal displacement $y(x,t)$ of the bar satisfies the wave equation with some wave speed c:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

for $x \in (0,\pi)$ and $t > 0$ with boundary conditions:

$$\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(\pi,t) = 0,$$

for $t > 0$. The bar is initially at rest so that

$$\frac{\partial y}{\partial t}(x,0) = 0$$

for $x \in (0,\pi)$, with a spatially varying initial longitudinal displacement given by

$$y(x,0) = bx$$

for $x \in (0,\pi)$, where b is a real constant.

(a) Using separation of variables, show that

$$y(x,t) = \frac{b\pi}{2} - \frac{4b}{\pi} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)x]\cos[(2n-1)ct]}{(2n-1)^2}.$$

(b) Determine a periodic function $P(x)$ such that this solution may be expressed as

$$y(x,t) = \frac{1}{2}[P(x+ct) + P(x-ct)].$$

Sketch $P(x)$ for $x \in [-3\pi, 3\pi]$.

17B Quantum Mechanics

The one dimensional quantum harmonic oscillator has Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2,$$

where m and ω are real positive constants and $\hat{x}$ and $\hat{p}$ are the standard position and momentum operators satisfying the commutation relation $[\hat{x},\hat{p}] = i\hbar$. Consider the operators

$$\hat{A} = \hat{p} - im\omega\hat{x} \quad \text{and} \quad \hat{B} = \hat{p} + im\omega\hat{x}.$$

(a) Show that

$$\hat{B}\hat{A} = 2m\left(\hat{H} - \frac{1}{2}\hbar\omega\right) \quad \text{and} \quad \hat{A}\hat{B} = 2m\left(\hat{H} + \frac{1}{2}\hbar\omega\right).$$

(b) Suppose that ϕ is an eigenfunction of $\hat{H}$ with eigenvalue E. Show that $\hat{A}\phi$ is then also an eigenfunction of $\hat{H}$ and that its corresponding eigenvalue is $E - \hbar\omega$.

(c) Show that for any normalisable wavefunctions χ and ψ,

$$\int_{-\infty}^{\infty} \chi^* \,(\hat{A}\psi)\, dx = \int_{-\infty}^{\infty} (\hat{B}\chi)^* \,\psi\, dx.$$

[You may assume that the operators $\hat{x}$ and $\hat{p}$ are Hermitian.]

(d) With ϕ as in (b), obtain an expression for the norm of $\hat{A}\phi$ in terms of E and the norm of ϕ. [The squared norm of any wavefunction ψ is $\int_{-\infty}^{\infty} |\psi|^2\, dx$.]

(e) Show that all eigenvalues of $\hat{H}$ are non-negative.

(f) Using the above results, deduce that each eigenvalue E of $\hat{H}$ must be of the form $E = (n + \frac{1}{2})\hbar\omega$ for some non-negative integer n.

18D Electromagnetism

(a) State the covariant form of Maxwell's equations and define all the quantities that appear in these expressions.

(b) Show that $\mathbf{E} \cdot \mathbf{B}$ is a Lorentz scalar (invariant under Lorentz transformations) and find another Lorentz scalar involving $\mathbf{E}$ and $\mathbf{B}$.

(c) In some inertial frame S the electric and magnetic fields are respectively $\mathbf{E} = (0, E_y, E_z)$ and $\mathbf{B} = (0, B_y, B_z)$. Find the electric and magnetic fields, $\mathbf{E}' = (0, E'_y, E'_z)$ and $\mathbf{B}' = (0, B'_y, B'_z)$, in another inertial frame S' that is related to S by the Lorentz transformation,

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where v is the velocity of S' in S and $\gamma = (1 - v^2/c^2)^{-1/2}$.

(d) Suppose that $\mathbf{E} = E_0(0, 1, 0)$ and $\mathbf{B} = \frac{E_0}{c}(0, \cos\theta, \sin\theta)$ where $0 \leqslant \theta \leqslant \pi/2$, and E_0 is a real constant. An observer is moving in S with velocity v parallel to the x-axis. What must v be for the electric and magnetic fields to appear to the observer to be parallel? Comment on the case $\theta = \pi/2$.

19D Numerical Analysis

(a) Define a *Givens rotation* $\Omega^{[p,q]} \in \mathbb{R}^{m \times m}$ and show that it is an orthogonal matrix.

(b) Define a QR *factorization* of a matrix $\mathsf{A} \in \mathbb{R}^{m \times n}$ with $m \geqslant n$. Explain how Givens rotations can be used to find $\mathsf{Q} \in \mathbb{R}^{m \times m}$ and $\mathsf{R} \in \mathbb{R}^{m \times n}$.

(c) Let

$$\mathsf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3/4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 98/25 \\ 25 \\ 25 \\ 0 \end{bmatrix}.$$

(i) Find a QR factorization of A using Givens rotations.

(ii) Hence find the vector $\mathbf{x}^* \in \mathbb{R}^3$ which minimises $\|\mathsf{A}\mathbf{x} - \mathbf{b}\|$, where $\|\cdot\|$ is the Euclidean norm. What is $\|\mathsf{A}\mathbf{x}^* - \mathbf{b}\|$?

20H Markov Chains

(a) Prove that every open communicating class of a Markov chain is transient. Prove that every finite transient communicating class is open. Give an example of a Markov chain with an infinite transient closed communicating class.

(b) Consider a Markov chain $(X_n)_{n\geqslant 0}$ with state space $\{a,b,c,d\}$ and transition probabilities given by the matrix

$$P=\begin{pmatrix} 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/4 & 0 & 3/4 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 \end{pmatrix}.$$

(i) Compute $\mathbb{P}(X_n = b | X_0 = d)$ for a fixed $n \geqslant 0$.

(ii) Compute $\mathbb{P}(X_n = c \text{ for some } n \geqslant 1 | X_0 = a)$.

(iii) Show that P^n converges as $n \to \infty$, and determine the limit.
[Results from lectures can be used without proof if stated carefully.]

END OF PAPER