UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday, 31 May, 2016 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

(a) Consider the linear transformation $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$\begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}.$$

Find a basis of $\mathbb{R}^3$ in which α is represented by a diagonal matrix.

(b) Give a list of 6×6 matrices such that any linear transformation $\beta : \mathbb{R}^6 \to \mathbb{R}^6$ with characteristic polynomial

$$(x-2)^4(x+7)^2$$

and minimal polynomial

$$(x-2)^2(x+7)$$

is similar to one of the matrices on your list. No two distinct matrices on your list should be similar. [No proof is required.]

2A Complex Analysis or Complex Methods

Classify the singularities of the following functions at both $z = 0$ and at the point at infinity on the extended complex plane:

$$\begin{aligned} f_1(z) &= \frac{e^z}{z\sin^2 z}, \\ f_2(z) &= \frac{1}{z^2(1-\cos z)}, \\ f_3(z) &= z^2\sin(1/z). \end{aligned}$$

3F Geometry

(a) Describe the Poincaré disc model D for the hyperbolic plane by giving the appropriate Riemannian metric.

(b) Let $a \in D$ be some point. Write down an isometry $f : D \to D$ with $f(a) = 0$.

(c) Using the Poincaré disc model, calculate the distance from 0 to $re^{i\theta}$ with $0 \leqslant r < 1$.

(d) Using the Poincaré disc model, calculate the area of a disc centred at a point $a \in D$ and of hyperbolic radius $\rho > 0$.

4C Variational Principles

(a) Consider the function $f(x_1, x_2) = 2x_1^2 + x_2^2 + \alpha x_1 x_2$, where α is a real constant. For what values of α is the function f convex?

(b) In the case $\alpha = -3$, calculate the extremum of x_1^2 on the set of points where $f(x_1, x_2) + 1 = 0$.

5C Fluid Dynamics

Consider the flow field in cartesian coordinates (x, y, z) given by

$$\mathbf{u} = \left(-\frac{Ay}{x^2 + y^2}, \frac{Ax}{x^2 + y^2}, U(z) \right),$$

where A is a constant. Let $\mathcal{D}$ denote the whole of $\mathbb{R}^3$ excluding the z axis.

(a) Determine the conditions on A and $U(z)$ for the flow to be both incompressible and irrotational in $\mathcal{D}$.

(b) Calculate the circulation along any closed curve enclosing the z axis.

6D Numerical Analysis

(a) What are real *orthogonal polynomials* defined with respect to an inner product $\langle \cdot, \cdot \rangle$? What does it mean for such polynomials to be *monic*?

(b) Real monic orthogonal polynomials, $p_n(x)$, of degree $n = 0, 1, 2, \ldots$, are defined with respect to the inner product,

$$\langle p, q \rangle = \int_{-1}^{1} w(x) p(x) q(x)\, dx,$$

where $w(x)$ is a positive weight function. Show that such polynomials obey the three-term recurrence relation,

$$p_{n+1}(x) = (x - \alpha_n) p_n(x) - \beta_n p_{n-1}(x),$$

for appropriate α_n and β_n which should be given in terms of inner products.

7H Statistics

Let $X_1, \ldots, X_n$ be independent samples from the exponential distribution with density $f(x; \lambda) = \lambda e^{-\lambda x}$ for $x > 0$, where λ is an unknown parameter. Find the critical region of the most powerful test of size α for the hypotheses $H_0 : \lambda = 1$ versus $H_1 : \lambda = 2$. Determine whether or not this test is uniformly most powerful for testing $H_0' : \lambda \leqslant 1$ versus $H_1' : \lambda > 1$.

8H Optimization

Let

$$A = \begin{pmatrix} 5 & -2 & -5 \\ -2 & 3 & 2 \\ -3 & 6 & 2 \\ 4 & -8 & -6 \end{pmatrix}$$

be the payoff of a two-person zero-sum game, where player I (randomly) picks a row to maximise the expected payoff and player II picks a column to minimise the expected payoff. Find each player's optimal strategy and the value of the game.

SECTION II

9F Linear Algebra

Let $M_{n,n}$ denote the vector space over $F = \mathbb{R}$ or $\mathbb{C}$ of $n \times n$ matrices with entries in F. Let $\mathrm{Tr} : M_{n,n} \to F$ denote the trace functional, i.e., if $A = (a_{ij})_{1 \leqslant i,j \leqslant n} \in M_{n,n}$, then

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

(a) Show that Tr is a linear functional.

(b) Show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for $A, B \in M_{n,n}$.

(c) Show that Tr is unique in the following sense: If $f : M_{n,n} \to F$ is a linear functional such that $f(AB) = f(BA)$ for each $A, B \in M_{n,n}$, then f is a scalar multiple of the trace functional. If, in addition, $f(I) = n$, then $f = \mathrm{Tr}$.

(d) Let $W \subseteq M_{n,n}$ be the subspace spanned by matrices C of the form $C = AB - BA$ for $A, B \in M_{n,n}$. Show that W is the kernel of Tr.

10E Groups, Rings and Modules

(a) Let I be an ideal of a commutative ring R and assume $I \subseteq \bigcup_{i=1}^{n} P_i$ where the P_i are prime ideals. Show that $I \subseteq P_i$ for some i.

(b) Show that $(x^2 + 1)$ is a maximal ideal of $\mathbb{R}[x]$. Show that the quotient ring $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{C}$.

(c) For $a, b \in \mathbb{R}$, let $I_{a,b}$ be the ideal $(x - a, y - b)$ in $\mathbb{R}[x, y]$. Show that $I_{a,b}$ is a maximal ideal. Find a maximal ideal J of $\mathbb{R}[x, y]$ such that $J \neq I_{a,b}$ for any $a, b \in \mathbb{R}$. Justify your answers.

11G Analysis II

Let (X, d) be a metric space.

(a) What does it mean to say that $(x_n)_n$ is a *Cauchy sequence* in X? Show that if $(x_n)_n$ is a Cauchy sequence, then it converges if it contains a convergent subsequence.

(b) Let $(x_n)_n$ be a Cauchy sequence in X.

(i) Show that for every $m \geqslant 1$, the sequence $(d(x_m, x_n))_n$ converges to some $d_m \in \mathbb{R}$.

(ii) Show that $d_m \to 0$ as $m \to \infty$.

(iii) Let $(y_n)_n$ be a subsequence of $(x_n)_n$. If ℓ, m are such that $y_\ell = x_m$, show that $d(y_\ell, y_n) \to d_m$ as $n \to \infty$.

(iv) Show also that for every m and n,

$$d_m - d_n \leqslant d(x_m, x_n) \leqslant d_m + d_n.$$

(v) Deduce that $(x_n)_n$ has a subsequence $(y_n)_n$ such that for every m and n,

$$d(y_{m+1}, y_m) \leqslant \frac{1}{3} d(y_m, y_{m-1})$$

and

$$d(y_{m+1}, y_{n+1}) \leqslant \frac{1}{2} d(y_m, y_n).$$

(c) Suppose that every closed subset Y of X has the property that every contraction mapping $Y \to Y$ has a fixed point. Prove that X is complete.

12E Metric and Topological Spaces

Let p be a prime number. Define what is meant by the *p-adic metric* d_p on $\mathbb{Q}$. Show that for $a, b, c \in \mathbb{Q}$ we have

$$d_p(a, b) \leqslant \max\{d_p(a, c), d_p(c, b)\}.$$

Show that the sequence (a_n), where $a_n = 1 + p + \cdots + p^{n-1}$, converges to some element in $\mathbb{Q}$.

For $a \in \mathbb{Q}$ define $|a|_p = d_p(a, 0)$. Show that if $a, b \in \mathbb{Q}$ and if $|a|_p \neq |b|_p$, then

$$|a + b|_p = \max\{|a|_p, |b|_p\}.$$

Let $a \in \mathbb{Q}$ and let $B(a, \delta)$ be the open ball with centre a and radius $\delta > 0$, with respect to the metric d_p. Show that $B(a, \delta)$ is a closed subset of $\mathbb{Q}$ with respect to the topology induced by d_p.

13A Complex Analysis or Complex Methods

Let $w = u + iv$ and let $z = x + iy$, for u, v, x, y real.

(a) Let A be the map defined by $w = \sqrt{z}$, using the principal branch. Show that A maps the region to the left of the parabola $y^2 = 4(1 - x)$ on the $z-$plane, with the negative real axis $x \in (-\infty, 0]$ removed, into the vertical strip of the $w-$plane between the lines $u = 0$ and $u = 1$.

(b) Let B be the map defined by $w = \tan^2(z/2)$. Show that B maps the vertical strip of the $z-$plane between the lines $x = 0$ and $x = \pi/2$ into the region inside the unit circle on the $w-$plane, with the part $u \in (-1, 0]$ of the negative real axis removed.

(c) Using the results of parts (a) and (b), show that the map C, defined by $w = \tan^2(\pi\sqrt{z}/4)$, maps the region to the left of the parabola $y^2 = 4(1 - x)$ on the $z-$plane, *including* the negative real axis, onto the unit disc on the $w-$plane.

14A Methods

(a) Consider the general self-adjoint problem for $y(x)$ on $[a, b]$:

$$-\frac{d}{dx}\left[p(x)\frac{d}{dx}y\right] + q(x)y = \lambda w(x)y; \quad y(a) = y(b) = 0,$$

where λ is the eigenvalue, and $w(x) > 0$. Prove that eigenfunctions associated with distinct eigenvalues are orthogonal with respect to a particular inner product which you should define carefully.

(b) Consider the problem for $y(x)$ given by

$$xy'' + 3y' + \left(\frac{1+\lambda}{x}\right)y = 0; \quad y(1) = y(e) = 0.$$

(i) Recast this problem into self-adjoint form.

(ii) Calculate the complete set of eigenfunctions and associated eigenvalues for this problem. [*Hint: You may find it useful to make the substitution* $x = e^s$.]

(iii) Verify that the eigenfunctions associated with distinct eigenvalues are indeed orthogonal.

15B Quantum Mechanics

(a) A particle of mass m in one space dimension is confined to move in a potential $V(x)$ given by

$$V(x)=\begin{cases} 0 & \text{for } 0<x<a, \\ \infty & \text{for } x<0 \text{ or } x>a. \end{cases}$$

The normalised initial wavefunction of the particle at time $t=0$ is

$$\psi_0(x)=\frac{4}{\sqrt{5a}}\sin^3\left(\frac{\pi x}{a}\right).$$

(i) Find the expectation value of the energy at time $t=0$.

(ii) Find the wavefunction of the particle at time $t=1$.

[*Hint: It may be useful to recall the identity* $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.]

(b) The right hand wall of the potential is lowered to a finite constant value $U_0>0$ giving the new potential:

$$U(x)=\begin{cases} 0 & \text{for } 0<x<a, \\ \infty & \text{for } x<0, \\ U_0 & \text{for } x>a. \end{cases}$$

This potential is set up in the laboratory but the value of U_0 is unknown. The stationary states of the potential are investigated and it is found that there exists exactly one bound state. Show that the value of U_0 must satisfy

$$\frac{\pi^2\hbar^2}{8ma^2}<U_0<\frac{9\pi^2\hbar^2}{8ma^2}.$$

16D Electromagnetism

(a) From the differential form of Maxwell's equations with $\mathbf{J} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$ and a time-independent electric field, derive the integral form of Gauss's law.

(b) Derive an expression for the electric field $\mathbf{E}$ around an infinitely long line charge lying along the z-axis with charge per unit length μ. Find the electrostatic potential ϕ up to an arbitrary constant.

(c) Now consider the line charge with an ideal earthed conductor filling the region $x > d$. State the boundary conditions satisfied by ϕ and $\mathbf{E}$ on the surface of the conductor.

(d) Show that the same boundary conditions at $x = d$ are satisfied if the conductor is replaced by a second line charge at $x = 2d$, $y = 0$ with charge per unit length $-\mu$.

(e) Hence or otherwise, returning to the setup in (c), calculate the force per unit length acting on the line charge.

(f) What is the charge per unit area $\sigma(y, z)$ on the surface of the conductor?

17C Fluid Dynamics

(a) For a velocity field $\mathbf{u}$, show that $\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}\mathbf{u}^2\right) - \mathbf{u} \times \boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the flow vorticity.

(b) For a scalar field $H(\mathbf{r})$, show that if $\mathbf{u} \cdot \boldsymbol{\nabla} H = 0$, then H is constant along the flow streamlines.

(c) State the Euler equations satisfied by an inviscid fluid of constant density subject to conservative body forces.

(i) If the flow is irrotational, show that an exact first integral of the Euler equations may be obtained.

(ii) If the flow is not irrotational, show that although an exact first integral of the Euler equations may not be obtained, a similar quantity is constant along the flow streamlines provided the flow is steady.

(iii) If the flow is now in a frame rotating with steady angular velocity $\Omega \mathbf{e}_z$, establish that a similar quantity is constant along the flow streamlines with an extra term due to the centrifugal force when the flow is steady.

18D Numerical Analysis

(a) Consider a method for numerically solving an ordinary differential equation (ODE) for an initial value problem, $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$. What does it mean for a method to *converge* over $t \in [0, T]$ where $T \in \mathbb{R}$? What is the definition of the *order* of a method?

(b) A general multistep method for the numerical solution of an ODE is

$$\sum_{l=0}^{s} \rho_l \, \mathbf{y}_{n+l} = h \sum_{l=0}^{s} \sigma_l \, \mathbf{f}(t_{n+l}, \mathbf{y}_{n+l}), \qquad n = 0, 1, \ldots ,$$

where s is a fixed positive integer. Show that this method is at least of order $p \geqslant 1$ if and only if

$$\sum_{l=0}^{s} \rho_l = 0 \qquad \text{and} \qquad \sum_{l=0}^{s} l^k \, \rho_l = k \sum_{l=0}^{s} l^{k-1} \, \sigma_l, \qquad k = 1, \ldots, p\,.$$

(c) State the Dahlquist equivalence theorem regarding the convergence of a multistep method.

(d) Consider the multistep method,

$$\mathbf{y}_{n+2} + \theta \, \mathbf{y}_{n+1} + a \, \mathbf{y}_n = h\big[\sigma_0 \mathbf{f}(t_n, \mathbf{y}_n) + \sigma_1 \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) + \sigma_2 \mathbf{f}(t_{n+2}, \mathbf{y}_{n+2})\big]\,.$$

Determine the values of σ_i and a (in terms of the real parameter θ) such that the method is at least third order. For what values of θ does the method converge?

19H Statistics

(a) What does it mean to say a statistic T is *sufficient* for an unknown parameter θ? State the factorisation criterion for sufficiency and prove it in the discrete case.

(b) State and prove the Rao-Blackwell theorem.

(c) Let $X_1, \ldots, X_n$ be independent samples from the uniform distribution on $[-\theta, \theta]$ for an unknown positive parameter θ. Consider the two-dimensional statistic

$$T = (\min_i X_i, \max_i X_i).$$

Prove that T is sufficient for θ. Determine, with proof, whether or not T is minimally sufficient.

20H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a simple symmetric random walk on the integers, starting at $X_0 = 0$.

(a) What does it mean to say that a Markov chain is *irreducible*? What does it mean to say that an irreducible Markov chain is *recurrent*? Show that $(X_n)_{n\geqslant 0}$ is irreducible and recurrent.

[*Hint: You may find it helpful to use the limit*

$$\lim_{k\to\infty} \sqrt{k}2^{-2k}\binom{2k}{k} = \sqrt{\pi}.$$

You may also use without proof standard necessary and sufficient conditions for recurrence.]

(b) What does it mean to say that an irreducible Markov chain is *positive recurrent*? Determine, with proof, whether $(X_n)_{n\geqslant 0}$ is positive recurrent.

(c) Let

$$T = \inf\{n \geqslant 1 : X_n = 0\}$$

be the first time the chain returns to the origin. Compute $\mathbb{E}[s^T]$ for a fixed number $0 < s < 1$.

END OF PAPER