UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Wednesday, 1 June, 2016 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

*The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt **all four** questions from Section I and **at most five** questions from Section II. In Section II, **no more than three** questions on each course may be attempted.*

Complete answers are preferred to fragments.

*Write on **one** side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in separate bundles, marked **A**, **B**, **C**, **D**, **E** and **F** according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

Find a pair of integers x and y satisfying $17x + 29y = 1$. What is the smallest positive integer congruent to 17^{138} modulo 29?

2E Numbers and Sets

Explain the meaning of the phrase *least upper bound;* state the least upper bound property of the real numbers. Use the least upper bound property to show that a bounded, increasing sequence of real numbers converges.

Suppose that $a_n, b_n \in \mathbb{R}$ and that $a_n \geqslant b_n > 0$ for all n. If $\sum_{n=1}^{\infty} a_n$ converges, show that $\sum_{n=1}^{\infty} b_n$ converges.

3B Dynamics and Relativity

With the help of definitions or equations of your choice, determine the dimensions, in terms of mass (M), length (L), time (T) and charge (Q), of the following quantities:

(i) force;

(ii) moment of a force (*i.e.* torque);

(iii) energy;

(iv) Newton's gravitational constant G;

(v) electric field $\mathbf{E}$;

(vi) magnetic field $\mathbf{B}$;

(vii) the vacuum permittivity ϵ_0.

4B Dynamics and Relativity

The radial equation of motion of a particle moving under the influence of a central force is

$$\ddot{r} - \frac{h^2}{r^3} = -kr^n,$$

where h is the angular momentum per unit mass of the particle, n is a constant, and k is a positive constant.

Show that circular orbits with $r = a$ are possible for any positive value of a, and that they are stable to small perturbations that leave h unchanged if $n > -3$.

SECTION II

5E Numbers and Sets

(a) Let S be a set. Show that there is no bijective map from S to the power set of S. Let $\mathcal{T} = \{(x_n) \mid x_i \in \{0,1\} \text{ for all } i \in \mathbb{N}\}$ be the set of sequences with entries in $\{0,1\}$. Show that $\mathcal{T}$ is uncountable.

(b) Let A be a finite set with more than one element, and let B be a countably infinite set. Determine whether each of the following sets is countable. Justify your answers.

(i) $S_1 = \{f : A \to B \mid f \text{ is injective}\}$.

(ii) $S_2 = \{g : B \to A \mid g \text{ is surjective}\}$.

(iii) $S_3 = \{h : B \to B \mid h \text{ is bijective}\}$.

6E Numbers and Sets

Suppose that $a, b \in \mathbb{Z}$ and that $b = b_1 b_2$, where b_1 and b_2 are relatively prime and greater than 1. Show that there exist unique integers $a_1, a_2, n \in \mathbb{Z}$ such that $0 \leqslant a_i < b_i$ and

$$\frac{a}{b} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + n.$$

Now let $b = p_1^{n_1} \ldots p_k^{n_k}$ be the prime factorization of b. Deduce that $\dfrac{a}{b}$ can be written uniquely in the form

$$\frac{a}{b} = \frac{q_1}{p_1^{n_1}} + \cdots + \frac{q_k}{p_k^{n_k}} + n\,,$$

where $0 \leqslant q_i < p_i^{n_i}$ and $n \in \mathbb{Z}$. Express $\dfrac{a}{b} = \dfrac{1}{315}$ in this form.

7E Numbers and Sets

State the inclusion-exclusion principle.

Let $A = (a_1, a_2, \ldots, a_n)$ be a string of n digits, where $a_i \in \{0, 1, \ldots, 9\}$. We say that the string A has a run of length k if there is some $j \leqslant n - k + 1$ such that either $a_{j+i} \equiv a_j + i \pmod{10}$ for all $0 \leqslant i < k$ or $a_{j+i} \equiv a_j - i \pmod{10}$ for all $0 \leqslant i < k$. For example, the strings

$$(\underline{0,1,2},8,4,9),\ \ (3,\underline{9,8,7},4,8) \text{ and } (3,\underline{1,0,9},4,5)$$

all have runs of length 3 (underlined), but no run in $(3,1,2,1,1,2)$ has length > 2. How many strings of length 6 have a run of length $\geqslant 3$?

8E Numbers and Sets

Define the binomial coefficient $\binom{n}{m}$. Prove directly from your definition that

$$(1+z)^n = \sum_{m=0}^{n} \binom{n}{m} z^m$$

for any complex number z.

(a) Using this formula, or otherwise, show that

$$\sum_{k=0}^{3n} (-3)^k \binom{6n}{2k} = 2^{6n}.$$

(b) By differentiating, or otherwise, evaluate $\sum_{m=0}^{n} m \binom{n}{m}$.

Let $S_r(n) = \sum_{m=0}^{n} (-1)^m m^r \binom{n}{m}$, where r is a non-negative integer. Show that $S_r(n) = 0$ for $r < n$. Evaluate $S_n(n)$.

9B Dynamics and Relativity

(a) A rocket, moving non-relativistically, has speed $v(t)$ and mass $m(t)$ at a time t after it was fired. It ejects mass with constant speed u relative to the rocket. Let the total momentum, at time t, of the system (rocket and ejected mass) in the direction of the motion of the rocket be $P(t)$. Explain carefully why $P(t)$ can be written in the form

$$P(t) = m(t)\, v(t) - \int_0^t \big(v(\tau) - u\big) \frac{dm(\tau)}{d\tau} d\tau\,. \qquad (*)$$

If the rocket experiences no external force, show that

$$m\frac{dv}{dt} + u\frac{dm}{dt} = 0\,. \qquad (\dagger)$$

Derive the expression corresponding to $(*)$ for the total kinetic energy of the system at time t. Show that kinetic energy is not necessarily conserved.

(b) Explain carefully how $(*)$ should be modified for a rocket moving relativistically, given that there are no external forces. Deduce that

$$\frac{d(m\gamma v)}{dt} = \left(\frac{v-u}{1-uv/c^2}\right)\frac{d(m\gamma)}{dt}\,,$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$ and hence that

$$m\gamma^2\frac{dv}{dt} + u\frac{dm}{dt} = 0\,. \qquad (\ddagger)$$

(c) Show that $(\dagger)$ and $(\ddagger)$ agree in the limit $c \to \infty$. Briefly explain the fact that kinetic energy is not conserved for the non-relativistic rocket, but relativistic energy is conserved for the relativistic rocket.

10B Dynamics and Relativity

A particle of unit mass moves with angular momentum h in an attractive central force field of magnitude $\dfrac{k}{r^2}$, where r is the distance from the particle to the centre and k is a constant. *You may assume* that the equation of its orbit can be written in plane polar coordinates in the form

$$r = \frac{\ell}{1 + e\cos\theta}\,,$$

where $\ell = \dfrac{h^2}{k}$ and e is the eccentricity. Show that the energy of the particle is

$$\frac{h^2(e^2-1)}{2\ell^2}\,.$$

A comet moves in a parabolic orbit about the Sun. When it is at its perihelion, a distance d from the Sun, and moving with speed V, it receives an impulse which imparts an additional velocity of magnitude αV directly away from the Sun. Show that the eccentricity of its new orbit is $\sqrt{1+4\alpha^2}$, and sketch the two orbits on the same axes.

11B Dynamics and Relativity

(a) Alice travels at constant speed v to Alpha Centauri, which is at distance d from Earth. She then turns around (taking very little time to do so), and returns at speed v. Bob stays at home. By how much has Bob aged during the journey? By how much has Alice aged? [No justification is required.]

Briefly explain what is meant by the *twin paradox* in this context. Why is it not a paradox?

(b) Suppose instead that Alice's world line is given by

$$-c^2t^2 + x^2 = c^2t_0^2\,,$$

where t_0 is a positive constant. Bob stays at home, at $x = \alpha c t_0$, where $\alpha > 1$. Alice and Bob compare their ages on both occasions when they meet. By how much does Bob age? Show that Alice ages by $2t_0\cosh^{-1}\alpha$.

12B Dynamics and Relativity

State what the vectors $\mathbf{a}$, $\mathbf{r}$, $\mathbf{v}$ and $\boldsymbol{\omega}$ represent in the following equation:

$$\mathbf{a} = \mathbf{g} - 2\,\boldsymbol{\omega} \times \mathbf{v} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,, \qquad (*)$$

where $\mathbf{g}$ is the acceleration due to gravity.

Assume that the radius of the Earth is 6×10^6 m, that $|\mathbf{g}| = 10\,\mathrm{ms}^{-2}$, and that there are 9×10^4 seconds in a day. Use these data to determine roughly the order of magnitude of each term on the right hand side of $(*)$ in the case of a particle dropped from a point at height 20 m above the surface of the Earth.

Taking again $|\mathbf{g}| = 10\,\mathrm{ms}^{-2}$, find the time T of the particle's fall in the absence of rotation.

Use a suitable approximation scheme to show that

$$\mathbf{R} \approx \mathbf{R}_0 - \frac{1}{3}\boldsymbol{\omega} \times \mathbf{g}\,T^3 - \frac{1}{2}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}_0)\,T^2\,,$$

where $\mathbf{R}$ is the position vector of the point at which the particle lands, and $\mathbf{R}_0$ is the position vector of the point at which the particle would have landed in the absence of rotation.

The particle is dropped at latitude 45°. Find expressions for the approximate northerly and easterly displacements of $\mathbf{R}$ from $\mathbf{R}_0$ in terms of ω, g, R_0 (the magnitudes of $\boldsymbol{\omega}$, $\mathbf{g}$ and $\mathbf{R}_0$, respectively), and T. You should ignore the curvature of the Earth's surface.

END OF PAPER