UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 30 May, 2016 9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

Let G be a group, and let H be a subgroup of G. Show that the following are equivalent.

(i) $a^{-1}b^{-1}ab \in H$ for all $a, b \in G$.

(ii) H is a normal subgroup of G and G/H is abelian.

Hence find all abelian quotient groups of the dihedral group D_{10} of order 10.

2D Groups

State and prove Lagrange's theorem.

Let p be an odd prime number, and let G be a finite group of order $2p$ which has a normal subgroup of order 2. Show that G is a cyclic group.

3C Vector Calculus

State the chain rule for the derivative of a composition $t \mapsto f(\mathbf{X}(t))$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{X} : \mathbb{R} \to \mathbb{R}^n$ are smooth.

Consider parametrized curves given by

$$\mathbf{x}(t) = (x(t), y(t)) = (a\cos t, a\sin t)\,.$$

Calculate the tangent vector $\dfrac{d\mathbf{x}}{dt}$ in terms of $x(t)$ and $y(t)$. Given that $u(x,y)$ is a smooth function in the upper half-plane $\{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$ satisfying

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = u\,,$$

deduce that

$$\frac{d}{dt}\,u\big(x(t), y(t)\big) \;=\; u\big(x(t), y(t)\big).$$

If $u(1,1) = 10$, find $u(-1,1)$.

4C Vector Calculus

If $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are vectors in $\mathbb{R}^3$, show that $T_{ij} = v_i w_j$ defines a rank 2 tensor. For which choices of the vectors $\mathbf{v}$ and $\mathbf{w}$ is T_{ij} isotropic?

Write down the most general isotropic tensor of rank 2.

Prove that ϵ_{ijk} defines an isotropic rank 3 tensor.

SECTION II

5D Groups

For each of the following, either give an example or show that none exists.

(i) A non-abelian group in which every non-trivial element has order 2.

(ii) A non-abelian group in which every non-trivial element has order 3.

(iii) An element of S_9 of order 18.

(iv) An element of S_9 of order 20.

(v) A finite group which is not isomorphic to a subgroup of an alternating group.

6D Groups

Define the *sign*, $\operatorname{sgn}(\sigma)$, of a permutation $\sigma \in S_n$ and prove that it is well defined. Show that the function $\operatorname{sgn} : S_n \to \{1, -1\}$ is a homomorphism.

Show that there is an injective homomorphism $\psi : GL_2(\mathbb{Z}/2\mathbb{Z}) \to S_4$ such that $\operatorname{sgn} \circ \psi$ is non-trivial.

Show that there is an injective homomorphism $\phi : S_n \to GL_n(\mathbb{R})$ such that $\det(\phi(\sigma)) = \operatorname{sgn}(\sigma)$.

7D Groups

State and prove the orbit-stabiliser theorem.

Let p be a prime number, and G be a finite group of order p^n with $n \geqslant 1$. If N is a non-trivial normal subgroup of G, show that $N \cap Z(G)$ contains a non-trivial element.

If H is a proper subgroup of G, show that there is a $g \in G \backslash H$ such that $g^{-1}Hg = H$.

[You may use Lagrange's theorem, provided you state it clearly.]

8D Groups

Define the *Möbius group* $\mathcal{M}$ and its action on the Riemann sphere $\mathbb{C}_\infty$. [You are not required to verify the group axioms.] Show that there is a surjective group homomorphism $\phi : SL_2(\mathbb{C}) \to \mathcal{M}$, and find the kernel of ϕ.

Show that if a non-trivial element of $\mathcal{M}$ has finite order, then it fixes precisely two points in $\mathbb{C}_\infty$. Hence show that any finite abelian subgroup of $\mathcal{M}$ is either cyclic or isomorphic to $C_2 \times C_2$.

[You may use standard properties of the Möbius group, provided that you state them clearly.]

9C Vector Calculus

What is a *conservative* vector field on $\mathbb{R}^n$?

State Green's theorem in the plane $\mathbb{R}^2$.

(a) Consider a smooth vector field $\mathbf{V} = \big(P(x,y), Q(x,y)\big)$ defined on all of $\mathbb{R}^2$ which satisfies

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0\,.$$

By considering

$$F(x,y) = \int_0^x P(x',0)\,dx' \;+\; \int_0^y Q(x,y')\,dy'$$

or otherwise, show that $\mathbf{V}$ is conservative.

(b) Now let $\mathbf{V} = \big(1+\cos(2\pi x + 2\pi y), 2 + \cos(2\pi x + 2\pi y)\big)$. Show that there exists a smooth function $F(x,y)$ such that $\mathbf{V} = \nabla F$.

Calculate $\int_C \mathbf{V}\cdot d\mathbf{x}$, where C is a smooth curve running from $(0,0)$ to $(m,n) \in \mathbb{Z}^2$. Deduce that there does *not* exist a smooth function $F(x,y)$ which satisfies $\mathbf{V} = \nabla F$ and which is, in addition, periodic with period 1 in each coordinate direction, *i.e.* $F(x,y) = F(x+1,y) = F(x,y+1)$.

10C Vector Calculus

Define the *Jacobian* $J[\mathbf{u}]$ of a smooth mapping $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$. Show that if $\mathbf{V}$ is the vector field with components

$$V_i = \frac{1}{3!}\epsilon_{ijk}\epsilon_{abc}\frac{\partial u_a}{\partial x_j}\,\frac{\partial u_b}{\partial x_k}\,u_c\,,$$

then $J[\mathbf{u}] = \nabla \cdot \mathbf{V}$. If $\mathbf{v}$ is another such mapping, state the chain rule formula for the derivative of the composition $\mathbf{w}(\mathbf{x}) = \mathbf{u}(\mathbf{v}(\mathbf{x}))$, and hence give $J[\mathbf{w}]$ in terms of $J[\mathbf{u}]$ and $J[\mathbf{v}]$.

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Let there be given, for each $t \in \mathbb{R}$, a smooth mapping $\mathbf{u}_t : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathbf{u}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{F}(\mathbf{x}) + o(t)$ as $t \to 0$. Show that

$$J[\mathbf{u}_t] \;=\; 1 + tQ(x) + o(t)$$

for some $Q(x)$, and express Q in terms of $\mathbf{F}$. Assuming now that $\mathbf{u}_{t+s}(\mathbf{x}) = \mathbf{u}_t(\mathbf{u}_s(\mathbf{x}))$, deduce that if $\nabla \cdot \mathbf{F} = 0$ then $J[\mathbf{u}_t] = 1$ for all $t \in \mathbb{R}$. What geometric property of the mapping $\mathbf{x} \mapsto \mathbf{u}_t(\mathbf{x})$ does this correspond to?

11C Vector Calculus

(a) For smooth scalar fields u and v, derive the identity

$$\nabla \cdot (u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u$$

and deduce that

$$\int_{\rho \leqslant |\mathbf{x}| \leqslant r} \left(v\nabla^2 u - u\nabla^2 v\right)\, dV = \int_{|\mathbf{x}|=r} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS - \int_{|\mathbf{x}|=\rho} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS\,.$$

Here ∇^2 is the Laplacian, $\dfrac{\partial}{\partial n} = \mathbf{n}\cdot\nabla$ where $\mathbf{n}$ is the unit outward normal, and dS is the scalar area element.

(b) Give the expression for $\left(\nabla \times \mathbf{V}\right)_i$ in terms of ϵ_{ijk}. Hence show that

$$\nabla \times \left(\nabla \times \mathbf{V}\right) \,=\, \nabla(\nabla \cdot \mathbf{V}) \,-\, \nabla^2 \mathbf{V}\,.$$

(c) Assume that if $\nabla^2\varphi = -\rho$, where $\varphi(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ and $\nabla\varphi(\mathbf{x}) = O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$, then

$$\varphi(\mathbf{x}) \,=\, \int_{\mathbb{R}^3} \frac{\rho(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\, dV\,.$$

The vector fields $\mathbf{B}$ and $\mathbf{J}$ satisfy

$$\nabla \times \mathbf{B} \,=\, \mathbf{J}\,.$$

Show that $\nabla \cdot \mathbf{J} = 0$. In the case that $\mathbf{B} = \nabla \times \mathbf{A}$, with $\nabla \cdot \mathbf{A} = 0$, show that

$$\mathbf{A}(\mathbf{x}) \,=\, \int_{\mathbb{R}^3} \frac{\mathbf{J}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\, dV\,, \qquad (*)$$

and hence that

$$\mathbf{B}(\mathbf{x}) \,=\, \int_{\mathbb{R}^3} \frac{\mathbf{J}(\mathbf{y}) \times (\mathbf{x}-\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^3}\, dV\,.$$

Verify that $\mathbf{A}$ given by $(*)$ does indeed satisfy $\nabla \cdot \mathbf{A} = 0$. [It may be useful to make a change of variables in the right hand side of $(*)$.]

12C Vector Calculus

(a) Let

$$\mathbf{F} = (z, x, y)$$

and let C be a circle of radius R lying in a plane with unit normal vector (a, b, c). Calculate $\nabla\times\mathbf{F}$ and use this to compute $\oint_C \mathbf{F}\cdot d\mathbf{x}$. Explain any orientation conventions which you use.

(b) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field such that the matrix with entries $\dfrac{\partial F_j}{\partial x_i}$ is symmetric. Prove that $\oint_C \mathbf{F}\cdot d\mathbf{x} = 0$ for every circle $C \subset \mathbb{R}^3$.

(c) Let $\mathbf{F} = \dfrac{1}{r}(x, y, z)$, where $r = \sqrt{x^2 + y^2 + z^2}$ and let C be the circle which is the intersection of the sphere $(x-5)^2+(y-3)^2+(z-2)^2 = 1$ with the plane $3x-5y-z = 2$. Calculate $\oint_C \mathbf{F}\cdot d\mathbf{x}$.

(d) Let $\mathbf{F}$ be the vector field defined, for $x^2 + y^2 > 0$, by

$$\mathbf{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z \right).$$

Show that $\nabla\times\mathbf{F} = \mathbf{0}$. Let C be the curve which is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $z = x + 200$. Calculate $\oint_C \mathbf{F}\cdot d\mathbf{x}$.

END OF PAPER