UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday, 27 May, 2016 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Differential Equations

(a) Find the solution of the differential equation

$$y'' - y' - 6y = 0$$

that is bounded as $x \to \infty$ and satisfies $y = 1$ when $x = 0$.

(b) Solve the difference equation

$$(y_{n+1} - 2y_n + y_{n-1}) - \frac{h}{2}(y_{n+1} - y_{n-1}) - 6h^2 y_n = 0.$$

Show that if $0 < h \ll 1$, the solution that is bounded as $n \to \infty$ and satisfies $y_0 = 1$ is approximately $(1 - 2h)^n$.

(c) By setting $x = nh$, explain the relation between parts (a) and (b).

2A Differential Equations

(a) For each non-negative integer n and positive constant λ, let

$$I_n(\lambda) = \int_0^\infty x^n e^{-\lambda x} dx.$$

By differentiating I_n with respect to λ, find its value in terms of n and λ.

(b) By making the change of variables $x = u + v,\ y = u - v$, transform the differential equation

$$\frac{\partial^2 f}{\partial x \partial y} = 1$$

into a differential equation for g, where $g(u, v) = f(x, y)$.

3F Probability

Let $X_1, \ldots, X_n$ be independent random variables, all with uniform distribution on $[0, 1]$. What is the probability of the event $\{X_1 > X_2 > \cdots > X_{n-1} > X_n\}$?

4F Probability

Define the *moment-generating function* m_Z of a random variable Z. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with distribution $\mathcal{N}(0,1)$, and let $Z = X_1^2 + \cdots + X_n^2$. For $\theta < 1/2$, show that

$$m_Z(\theta) = (1-2\theta)^{-n/2}\,.$$

SECTION II

5A Differential Equations

(a) Find and sketch the solution of

$$y'' + y = \delta(x - \pi/2),$$

where δ is the Dirac delta function, subject to $y(0) = 1$ and $y'(0) = 0$.

(b) A bowl of soup, which Sam has just warmed up, cools down at a rate equal to the product of a constant k and the difference between its temperature $T(t)$ and the temperature T_0 of its surroundings. Initially the soup is at temperature $T(0) = \alpha T_0$, where $\alpha > 2$.

(i) Write down and solve the differential equation satisfied by $T(t)$.

(ii) At time t_1, when the temperature reaches half of its initial value, Sam quickly adds some hot water to the soup, so the temperature increases instantaneously by β, where $\beta > \alpha T_0/2$. Find t_1 and $T(t)$ for $t > t_1$.

(iii) Sketch $T(t)$ for $t > 0$.

(iv) Sam wants the soup to be at temperature αT_0 at time t_2, where $t_2 > t_1$. What value of β should Sam choose to achieve this? Give your answer in terms of α, k, t_2 and T_0.

6A Differential Equations

(a) The function $y(x)$ satisfies

$$y'' + p(x)y' + q(x)y = 0.$$

(i) Define the *Wronskian* $W(x)$ of two linearly independent solutions $y_1(x)$ and $y_2(x)$. Derive a linear first-order differential equation satisfied by $W(x)$.

(ii) Suppose that $y_1(x)$ is known. Use the Wronskian to write down a first-order differential equation for $y_2(x)$. Hence express $y_2(x)$ in terms of $y_1(x)$ and $W(x)$.

(b) Verify that $y_1(x) = \cos(x^\gamma)$ is a solution of

$$ax^\alpha y'' + bx^{\alpha-1}y' + y = 0,$$

where a, b, α and γ are constants, provided that these constants satisfy certain conditions which you should determine.

Use the method that you described in part (a) to find a solution which is linearly independent of $y_1(x)$.

7A Differential Equations

The function $y(x)$ satisfies

$$y'' + p(x)y' + q(x)y = 0.$$

What does it mean to say that the point $x = 0$ is (i) an *ordinary point* and (ii) a *regular singular point* of this differential equation? Explain what is meant by the *indicial equation* at a regular singular point. What can be said about the nature of the solutions in the neighbourhood of a regular singular point in the different cases that arise according to the values of the roots of the indicial equation?

State the nature of the point $x = 0$ of the equation

$$xy'' + (x - m + 1)y' - (m - 1)y = 0. \qquad (*)$$

Set $y(x) = x^\sigma \sum_{n=0}^{\infty} a_n x^n$, where $a_0 \neq 0$, and find the roots of the indicial equation.

(a) Show that one solution of $(*)$ with $m \neq 0, -1, -2, \cdots$ is

$$y(x) = x^m \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n\ x^n}{(m+n)(m+n-1)\cdots(m+1)}\right),$$

and find a linearly independent solution in the case when m is not an integer.

(b) If m is a positive integer, show that $(*)$ has a polynomial solution.

(c) What is the form of the general solution of $(*)$ in the case $m = 0$? [You do not need to find the general solution explicitly.]

8A Differential Equations

(a) By considering eigenvectors, find the general solution of the equations

$$\begin{aligned}\frac{dx}{dt} &= 2x + 5y,\\ \frac{dy}{dt} &= -x - 2y,\end{aligned} \qquad (\dagger)$$

and show that it can be written in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix} + \beta \begin{pmatrix} 5\sin t \\ \cos t - 2\sin t \end{pmatrix},$$

where α and β are constants.

(b) For any square matrix M, $\exp(M)$ is defined by

$$\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!}.$$

Show that if M has constant elements, the vector equation $\dfrac{d\mathbf{x}}{dt} = M\mathbf{x}$ has a solution $\mathbf{x} = \exp(Mt)\mathbf{x}_0$, where $\mathbf{x}_0$ is a constant vector. Hence solve $(\dagger)$ and show that your solution is consistent with the result of part (a).

9F Probability

For any positive integer n and positive real number θ, the Gamma distribution $\Gamma(n,\theta)$ has density f_Γ defined on $(0,\infty)$ by

$$f_\Gamma(x) = \frac{\theta^n}{(n-1)!} x^{n-1} e^{-\theta x}.$$

For any positive integers a and b, the Beta distribution $B(a,b)$ has density f_B defined on $(0,1)$ by

$$f_B(x) = \frac{(a+b-1)!}{(a-1)!(b-1)!} x^{a-1}(1-x)^{b-1}.$$

Let X and Y be independent random variables with respective distributions $\Gamma(n,\theta)$ and $\Gamma(m,\theta)$. Show that the random variables $X/(X+Y)$ and $X+Y$ are independent and give their distributions.

10F Probability

We randomly place n balls in m bins independently and uniformly. For each i with $1 \leqslant i \leqslant m$, let B_i be the number of balls in bin i.

(a) What is the distribution of B_i? For $i \neq j$, are B_i and B_j independent?

(b) Let E be the number of empty bins, C the number of bins with two or more balls, and S the number of bins with exactly one ball. What are the expectations of E, C and S?

(c) Let $m = an$, for an integer $a \geqslant 2$. What is $\mathbb{P}(E = 0)$? What is the limit of $\mathbb{E}[E]/m$ when $n \to \infty$?

(d) Instead, let $n = dm$, for an integer $d \geqslant 2$. What is $\mathbb{P}(C = 0)$? What is the limit of $\mathbb{E}[C]/m$ when $n \to \infty$?

11F Probability

Let X be a non-negative random variable such that $\mathbb{E}[X^2] > 0$ is finite, and let $\theta \in [0, 1]$.

(a) Show that

$$\mathbb{E}[X\,\mathbb{I}[\{X > \theta\mathbb{E}[X]\}]] \geqslant (1 - \theta)\mathbb{E}[X]\,.$$

(b) Let Y_1 and Y_2 be random variables such that $\mathbb{E}[Y_1^2]$ and $\mathbb{E}[Y_2^2]$ are finite. State and prove the Cauchy–Schwarz inequality for these two variables.

(c) Show that

$$\mathbb{P}(X > \theta\,\mathbb{E}[X]) \geqslant (1 - \theta)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}\,.$$

12F Probability

A random graph with n nodes $v_1, \ldots, v_n$ is drawn by placing an edge with probability p between v_i and v_j for all distinct i and j, independently. A triangle is a set of three distinct nodes v_i, v_j, v_k that are all connected: there are edges between v_i and v_j, between v_j and v_k and between v_i and v_k.

(a) Let T be the number of triangles in this random graph. Compute the maximum value and the expectation of T.

(b) State the Markov inequality. Show that if $p = 1/n^{\alpha}$, for some $\alpha > 1$, then $\mathbb{P}(T = 0) \to 1$ when $n \to \infty$.

(c) State the Chebyshev inequality. Show that if p is such that $\mathrm{Var}[T]/\mathbb{E}[T]^2 \to 0$ when $n \to \infty$, then $\mathbb{P}(T = 0) \to 0$ when $n \to \infty$.

END OF PAPER