UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 26 May, 2016 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Vectors and Matrices

Let $z \in \mathbb{C}$ be a solution of

$$z^2 + bz + 1 = 0,$$

where $b \in \mathbb{R}$ and $|b| \leqslant 2$. For which values of b do the following hold?

(i) $|e^z| < 1$.

(ii) $|e^{iz}| = 1$.

(iii) $\mathrm{Im}(\cosh z) = 0$.

2C Vectors and Matrices

Write down the general form of a 2×2 rotation matrix. Let R be a real 2×2 matrix with positive determinant such that $|R\mathbf{x}| = |\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^2$. Show that R is a rotation matrix.

Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that any real 2×2 matrix A which satisfies $AJ = JA$ can be written as $A = \lambda R$, where λ is a real number and R is a rotation matrix.

3D Analysis I

What does it mean to say that a sequence of real numbers (x_n) converges to x? Suppose that (x_n) converges to x. Show that the sequence (y_n) given by

$$y_n = \frac{1}{n} \sum_{i=1}^{n} x_i$$

also converges to x.

4F Analysis I

Let a_n be the number of pairs of integers $(x, y) \in \mathbb{Z}^2$ such that $x^2 + y^2 \leqslant n^2$. What is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$? [You may use the comparison test, provided you state it clearly.]

SECTION II

5A Vectors and Matrices

(a) Use suffix notation to prove that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

(b) Show that the equation of the plane through three non-colinear points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}),$$

where $\mathbf{r}$ is the position vector of a point in this plane.

Find a unit vector normal to the plane in the case $\mathbf{a} = (2, 0, 1)$, $\mathbf{b} = (0, 4, 0)$ and $\mathbf{c} = (1, -1, 2)$.

(c) Let $\mathbf{r}$ be the position vector of a point in a given plane. The plane is a distance d from the origin and has unit normal vector $\mathbf{n}$, where $\mathbf{n} \cdot \mathbf{r} \geqslant 0$. Write down the equation of this plane.

This plane intersects the sphere with centre at $\mathbf{p}$ and radius q in a circle with centre at $\mathbf{m}$ and radius ρ. Show that

$$\mathbf{m} - \mathbf{p} = \gamma\, \mathbf{n}.$$

Find γ in terms of q and ρ. Hence find ρ in terms of $\mathbf{n}$, d, $\mathbf{p}$ and q.

6B Vectors and Matrices

The $n \times n$ real symmetric matrix M has eigenvectors of unit length $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Prove that the eigenvalues are real and that $\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$.

Let $\mathbf{x}$ be any (real) unit vector. Show that

$$\mathbf{x}^{\mathrm{T}} M \mathbf{x} \leqslant \lambda_1 .$$

What can be said about $\mathbf{x}$ if $\mathbf{x}^{\mathrm{T}} M \mathbf{x} = \lambda_1$?

Let S be the set of all (real) unit vectors of the form

$$\mathbf{x} = (0, x_2, \ldots, x_n) .$$

Show that $\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \in S$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Deduce that

$$\operatorname*{Max}_{\mathbf{x} \in S} \mathbf{x}^{\mathrm{T}} M \mathbf{x} \geqslant \lambda_2 .$$

The $(n-1) \times (n-1)$ matrix A is obtained by removing the first row and the first column of M. Let μ be the greatest eigenvalue of A. Show that

$$\lambda_1 \geqslant \mu \geqslant \lambda_2 .$$

7B Vectors and Matrices

What does it mean to say that a matrix can be diagonalised? Given that the $n \times n$ real matrix M has n eigenvectors satisfying $\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$, explain how to obtain the diagonal form Λ of M. Prove that Λ is indeed diagonal. Obtain, with proof, an expression for the trace of M in terms of its eigenvalues.

The elements of M are given by

$$M_{ij} = \begin{cases} 0 & \text{for } i = j , \\ 1 & \text{for } i \neq j . \end{cases}$$

Determine the elements of M^2 and hence show that, if λ is an eigenvalue of M, then

$$\lambda^2 = (n-1) + (n-2)\lambda .$$

Assuming that M can be diagonalised, give its diagonal form.

8C Vectors and Matrices

(a) Show that the equations

$$\begin{aligned} 1 + s + t &= a \\ 1 - s + t &= b \\ 1 - 2t &= c \end{aligned}$$

determine s and t uniquely if and only if $a + b + c = 3$.

Write the following system of equations

$$\begin{aligned} 5x + 2y - z &= 1 + s + t \\ 2x + 5y - z &= 1 - s + t \\ -x - y + 8z &= 1 - 2t \end{aligned}$$

in matrix form $A\mathbf{x} = \mathbf{b}$. Use Gaussian elimination to solve the system for x, y, and z. State a relationship between the rank and the kernel of a matrix. What is the rank and what is the kernel of A?

For which values of x, y, and z is it possible to solve the above system for s and t?

(b) Define a *unitary* $n \times n$ matrix. Let A be a real symmetric $n \times n$ matrix, and let I be the $n \times n$ identity matrix. Show that $|(A + iI)\mathbf{x}|^2 = |A\mathbf{x}|^2 + |\mathbf{x}|^2$ for arbitrary $\mathbf{x} \in \mathbb{C}^n$, where $|\mathbf{x}|^2 = \sum_{j=1}^{n} |x_j|^2$. Find a similar expression for $|(A - iI)\mathbf{x}|^2$. Prove that $(A - iI)(A + iI)^{-1}$ is well-defined and is a unitary matrix.

9E Analysis I

State the Bolzano–Weierstrass theorem. Use it to show that a continuous function $f : [a, b] \to \mathbb{R}$ attains a global maximum; that is, there is a real number $c \in [a, b]$ such that $f(c) \geqslant f(x)$ for all $x \in [a, b]$.

A function f is said to attain a local maximum at $c \in \mathbb{R}$ if there is some $\varepsilon > 0$ such that $f(c) \geqslant f(x)$ whenever $|x - c| < \varepsilon$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, and that $f''(x) < 0$ for all $x \in \mathbb{R}$. Show that there is at most one $c \in \mathbb{R}$ at which f attains a local maximum.

If there is a constant $K < 0$ such that $f''(x) < K$ for all $x \in \mathbb{R}$, show that f attains a global maximum. [*Hint: if $g'(x) < 0$ for all $x \in \mathbb{R}$, then g is decreasing.*]

Must $f : \mathbb{R} \to \mathbb{R}$ attain a global maximum if we merely require $f''(x) < 0$ for all $x \in \mathbb{R}$? Justify your answer.

10E Analysis I

Let $f : \mathbb{R} \to \mathbb{R}$. We say that $x \in \mathbb{R}$ is a real root of f if $f(x) = 0$. Show that if f is differentiable and has k distinct real roots, then f' has at least $k-1$ real roots. [Rolle's theorem may be used, provided you state it clearly.]

Let $p(x) = \sum_{i=1}^{n} a_i x^{d_i}$ be a polynomial in x, where all $a_i \neq 0$ and $d_{i+1} > d_i$. (In other words, the a_i are the nonzero coefficients of the polynomial, arranged in order of increasing power of x.) The *number of sign changes* in the coefficients of p is the number of i for which $a_i a_{i+1} < 0$. For example, the polynomial $x^5 - x^3 - x^2 + 1$ has 2 sign changes. Show by induction on n that the number of positive real roots of p is less than or equal to the number of sign changes in its coefficients.

11D Analysis I

If (x_n) and (y_n) are sequences converging to x and y respectively, show that the sequence $(x_n + y_n)$ converges to $x + y$.

If $x_n \neq 0$ for all n and $x \neq 0$, show that the sequence $\left(\dfrac{1}{x_n}\right)$ converges to $\dfrac{1}{x}$.

(a) Find $\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right)$.

(b) Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}}$ converges.

Justify your answers.

12F Analysis I

Let $f : [0,1] \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \in [0,1]$.

Show that f is continuous and that for all $\varepsilon > 0$, there exists a piecewise constant function g such that

$$\sup_{x\in[0,1]} |f(x) - g(x)| \leqslant \varepsilon.$$

For all integers $n \geqslant 1$, let $u_n = \int_0^1 f(t)\cos(nt)dt$. Show that the sequence (u_n) converges to 0.

END OF PAPER