UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday, 5 June, 2015 9:00 am to 12:00 noon

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *…,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Show that if $10^n + 1$ is prime then n must be a power of 2. Now assuming n is a power of 2, show that if p is a prime factor of $10^n + 1$ then $p \equiv 1 \pmod{2n}$.

Explain the method of Fermat factorization, and use it to factor $10^4 + 1$.

2I Topics in Analysis

Let $\mathcal{K}$ be the set of all non-empty compact subsets of m-dimensional Euclidean space $\mathbb{R}^m$. Define the Hausdorff metric on $\mathcal{K}$, and prove that it is a metric.

Let $K_1 \supseteq K_2 \supseteq \dots$ be a sequence in $\mathcal{K}$. Show that $K = \bigcap_{n=1}^{\infty} K_n$ is also in $\mathcal{K}$ and that $K_n \to K$ as $n \to \infty$ in the Hausdorff metric.

3G Coding and Cryptography

Explain how to construct binary Reed–Muller codes. State and prove a result giving the minimum distance for each such Reed–Muller code.

4J Statistical Modelling

Data on 173 nesting female horseshoe crabs record for each crab its colour as one of 4 factors (simply labelled $1, \ldots, 4$), its width (in cm) and the presence of male crabs nearby (a 1 indicating presence). The data are collected into the `R` data frame `crabs` and the first few lines are displayed below.

```
> crabs[1:4, ]
  colour width males
1      2  28.3     1
2      3  22.5     0
3      1  26.0     1
4      4  21.0     0
```

Describe the model being fitted by the `R` command below.

```
> fit1 <- glm(males ~ colour + width, family = binomial, data=crabs)
```

The following (abbreviated) output is obtained from the `summary` command.

```
> summary(fit1)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)    -11.38      2.873  -3.962 7.43e-05 ***
colour2          0.07      0.740   0.098    0.922
colour3         -0.22      0.777  -0.288    0.773
colour4         -1.32      0.853  -1.560    0.119
width            0.46      0.106   4.434 9.26e-06 ***
```

Write out the calculation for an approximate 95% confidence interval for the coefficient for `width`. Describe the calculation you would perform to obtain an estimate of the probability that a female crab of colour 3 and with a width of 20cm has males nearby. [You need not actually compute the end points of the confidence interval or the estimate of the probability above, but merely show the calculations that would need to be performed in order to arrive at them.]

5E Mathematical Biology

(i) A variant of the classic logistic population model is given by the Hutchinson–Wright equation

$$\frac{dx(t)}{dt} = \alpha\, x(t)\, [\, 1 - x(t-T)\,]$$

where $\alpha, T > 0$. Determine the condition on α (in terms of T) for the constant solution $x(t) = 1$ to be stable.

(ii) Another variant of the logistic model is given by the equation

$$\frac{dx(t)}{dt} = \alpha \left[\, x(t-T) - x(t)^2 \,\right] ,$$

where $\alpha, T > 0$. Give a brief interpretation of what this model represents.

Determine the condition on α (in terms of T) for the constant solution $x(t) = 1$ to be stable in this model.

6B Further Complex Methods

Explain how the Papperitz symbol

$$P \begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix}$$

represents a differential equation with certain properties. [You need not write down the differential equation explicitly.]

The hypergeometric function $F(a, b, c\,; z)$ is defined to be the solution of the equation given by the Papperitz symbol

$$P \begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix}$$

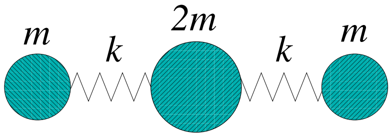
that is analytic at $z = 0$ and such that $F(a, b, c\,; 0) = 1$. Show that

$$F(a, b, c\,; z) = (1-z)^{-a} F\Big(a,\ c{-}b,\ c\,;\ \frac{z}{z-1}\Big) ,$$

indicating clearly any general results for manipulating Papperitz symbols that you use.

7D Classical Dynamics

A triatomic molecule is modelled by three masses moving in a line while connected to each other by two identical springs of force constant k as shown in the figure.



(a) Write down the Lagrangian and derive the equations describing the motion of the atoms.

(b) Find the normal modes and their frequencies. What motion does the lowest frequency represent?

8C Cosmology

Calculate the total effective number of relativistic spin states g_* present in the early universe when the temperature T is 10^{10} K if there are three species of low-mass neutrinos and antineutrinos in addition to photons, electrons and positrons. If the weak interaction rate is $\Gamma = (T/10^{10}\,\mathrm{K})^5\,\mathrm{s}^{-1}$ and the expansion rate of the universe is $H = \sqrt{8\pi G\rho/3}$, where ρ is the total density of the universe, calculate the temperature T_* at which the neutrons and protons cease to interact via weak interactions, and show that $T_* \propto g_*^{1/6}$.

State the formula for the equilibrium ratio of neutrons to protons at T_*, and briefly describe the sequence of events as the temperature falls from T_* to the temperature at which the nucleosynthesis of helium and deuterium ends.

What is the effect of an increase or decrease of g_* on the abundance of helium-4 resulting from nucleosynthesis? Why do changes in g_* have a very small effect on the final abundance of deuterium?

SECTION II

9H Number Theory

State the Chinese Remainder Theorem.

Let N be an odd positive integer. Define the Jacobi symbol $\left(\frac{a}{N}\right)$. Which of the following statements are true, and which are false? Give a proof or counterexample as appropriate.

(i) If $\left(\frac{a}{N}\right) = 1$ then the congruence $x^2 \equiv a \pmod{N}$ is soluble.

(ii) If N is not a square then $\sum_{a=1}^{N} \left(\frac{a}{N}\right) = 0$.

(iii) If N is composite then there exists an integer a coprime to N with

$$a^{N-1} \not\equiv 1 \pmod{N}.$$

(iv) If N is composite then there exists an integer a coprime to N with

$$a^{(N-1)/2} \not\equiv \left(\frac{a}{N}\right) \pmod{N}.$$

10J Statistical Modelling

Consider the normal linear model where the n-vector of responses Y satisfies $Y = X\beta + \varepsilon$ with $\varepsilon \sim N_n(0, \sigma^2 I)$. Here X is an $n \times p$ matrix of predictors with full column rank where $p \geqslant 3$ and $\beta \in \mathbb{R}^p$ is an unknown vector of regression coefficients. For $j \in \{1, \ldots, p\}$, denote the jth column of X by X_j, and let X_{-j} be X with its jth column removed. Suppose $X_1 = 1_n$ where 1_n is an n-vector of 1's. Denote the maximum likelihood estimate of β by $\hat{\beta}$. Write down the formula for $\hat{\beta}_j$ involving P_{-j}, the orthogonal projection onto the column space of X_{-j}.

Consider $j, k \in \{2, \ldots, p\}$ with $j < k$. By thinking about the orthogonal projection of X_j onto X_k, show that

$$\operatorname{var}(\hat{\beta}_j) \geqslant \frac{\sigma^2}{\|X_j\|^2} \left(1 - \left(\frac{X_k^T X_j}{\|X_k\| \|X_j\|} \right)^2 \right)^{-1}. \qquad (*)$$

[You may use standard facts about orthogonal projections including the fact that if V and W are subspaces of $\mathbb{R}^n$ with V a subspace of W and Π_V and Π_W denote orthogonal projections onto V and W respectively, then for all $v \in \mathbb{R}^n$, $\|\Pi_W v\|^2 \geqslant \|\Pi_V v\|^2$.]

This question continues on the next page

10J Statistical Modelling (continued)

By considering the fitted values $X\hat{\beta}$, explain why if, for any $j \geqslant 2$, a constant is added to each entry in the jth column of X, then $\hat{\beta}_j$ will remain unchanged. Let $\bar{X}_j = \sum_{i=1}^n X_{ij}/n$. Why is $(*)$ also true when all instances of X_j and X_k are replaced by $X_j - \bar{X}_j 1_n$ and $X_k - \bar{X}_k 1_n$ respectively?

The marks from mid-year statistics and mathematics tests and an end-of-year statistics exam are recorded for 100 secondary school students. The first few lines of the data are given below.

```
> exam_marks[1:3, ]
  Stat_exam Maths_test Stat_test
1        83         94        92
2        76         45        27
3        73         67        62
```

The following abbreviated output is obtained:

```
> summary(lm(Stat_exam ~ Maths_test + Stat_test, data=exam_marks))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  25.0342     8.2694   3.027  0.00316 **
Maths_test    0.2782     0.3708   0.750  0.45503
Stat_test     0.1643     0.3364   0.488  0.62641
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

F-statistic: 6.111 on 2 and 97 DF,  p-value: 0.003166
```

What are the hypothesis tests corresponding to the final column of the coefficients table? What is the hypothesis test corresponding to the final line of the output? Interpret the results when testing at the 5% level.

How does the following sample correlation matrix for the data help to explain the relative sizes of some of the p-values?

```
> cor(exam_marks)
           Stat_exam Maths_test Stat_test
Stat_exam  1.0000000   0.331224 0.3267138
Maths_test 0.3312240   1.000000 0.9371630
Stat_test  0.3267138   0.937163 1.0000000
```

11E Mathematical Biology

In a stochastic model of multiple populations, $P = P(\mathbf{x}, t)$ is the probability that the population sizes are given by the vector $\mathbf{x}$ at time t. The jump rate $W(\mathbf{x}, \mathbf{r})$ is the probability per unit time that the population sizes jump from $\mathbf{x}$ to $\mathbf{x} + \mathbf{r}$. Under suitable assumptions, the system may be approximated by the multivariate Fokker–Planck equation (with summation convention)

$$\frac{\partial}{\partial t} P = -\frac{\partial}{\partial x_i} A_i P + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} B_{ij} P \,,$$

where $A_i(\mathbf{x}) = \sum_{\mathbf{r}} r_i W(\mathbf{x}, \mathbf{r})$ and matrix elements $B_{ij}(\mathbf{x}) = \sum_{\mathbf{r}} r_i r_j W(\mathbf{x}, \mathbf{r})$.

(a) Use the multivariate Fokker–Planck equation to show that

$$\begin{aligned} \frac{d}{dt} \langle x_k \rangle &= \langle A_k \rangle \\ \frac{d}{dt} \langle x_k x_l \rangle &= \langle x_l A_k + x_k A_l + B_{kl} \rangle . \end{aligned}$$

[You may assume that $P(\mathbf{x}, t) \to 0$ as $|\mathbf{x}| \to \infty$.]

(b) For small fluctuations, you may assume that the vector $\mathbf{A}$ may be approximated by a linear function in $\mathbf{x}$ and the matrix $\mathbf{B}$ may be treated as constant, i.e. $A_k(\mathbf{x}) \approx a_{kl}(x_l - \langle x_l \rangle)$ and $B_{kl}(\mathbf{x}) \approx B_{kl}(\langle \mathbf{x} \rangle) = b_{kl}$ (where a_{kl} and b_{kl} are constants). Show that at steady state the covariances $C_{ij} = \operatorname{cov}(x_i, x_j)$ satisfy

$$a_{ik} C_{jk} + a_{jk} C_{ik} + b_{ij} = 0 \,.$$

(c) A lab-controlled insect population consists of x_1 larvae and x_2 adults. Larvae are added to the system at rate λ. Larvae each mature at rate γ per capita. Adults die at rate β per capita. Give the vector $\mathbf{A}$ and matrix $\mathbf{B}$ for this model. Show that at steady state

$$\langle x_1 \rangle = \frac{\lambda}{\gamma}, \quad \langle x_2 \rangle = \frac{\lambda}{\beta} \,.$$

(d) Find the variance of each population size near steady state, and show that the covariance between the populations is zero.

12C Classical Dynamics

Consider a rigid body with angular velocity $\boldsymbol{\omega}$, angular momentum $\mathbf{L}$ and position vector $\mathbf{r}$, in its body frame.

(a) Use the expression for the kinetic energy of the body,

$$\frac{1}{2}\int d^3\mathbf{r}\,\rho(\mathbf{r})\,\dot{\mathbf{r}}^2\,,$$

to derive an expression for the tensor of inertia of the body, $\mathbf{I}$. Write down the relationship between $\mathbf{L}$, $\mathbf{I}$ and $\boldsymbol{\omega}$.

(b) Euler's equations of torque-free motion of a rigid body are

$$\begin{aligned} I_1\,\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3\,, \\ I_2\,\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1\,, \\ I_3\,\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2\,. \end{aligned}$$

Working in the frame of the principal axes of inertia, use Euler's equations to show that the energy E and the squared angular momentum $\mathbf{L}^2$ are conserved.

(c) Consider a cuboid with sides a, b and c, and with mass M distributed uniformly.

(i) Use the expression for the tensor of inertia derived in (a) to calculate the principal moments of inertia of the body.

(ii) Assume $b = 2a$ and $c = 4a$, and suppose that the initial conditions are such that

$$\mathbf{L}^2 = 2I_2E$$

with the initial angular velocity $\boldsymbol{\omega}$ perpendicular to the intermediate principal axis $\mathbf{e}_2$. Derive the first order differential equation for ω_2 in terms of E, M and a and hence determine the long-term behaviour of $\boldsymbol{\omega}$.

13I Logic and Set Theory

State the Axiom of Foundation and the Principle of ϵ-Induction, and show that they are equivalent (in the presence of the other axioms of ZF). [You may assume the existence of transitive closures.]

Explain briefly how the Principle of ϵ-Induction implies that every set is a member of some V_α.

Find the ranks of the following sets:

(i) $\{\omega + 1, \omega + 2, \omega + 3\}$,

(ii) the Cartesian product $\omega \times \omega$,

(iii) the set of all functions from ω to ω^2.

[You may assume standard properties of rank.]

14I Graph Theory

Let G be a bipartite graph with vertex classes X and Y. What does it mean to say that G contains a *matching from X to Y*? State and prove Hall's Marriage Theorem.

Suppose now that every $x \in X$ has $d(x) \geqslant 1$, and that if $x \in X$ and $y \in Y$ with $xy \in E(G)$ then $d(x) \geqslant d(y)$. Show that G contains a matching from X to Y.

15F Representation Theory

(a) Let S^1 be the circle group. Assuming any required facts about continuous functions from real analysis, show that every 1-dimensional continuous representation of S^1 is of the form

$$z \mapsto z^n$$

for some $n \in \mathbb{Z}$.

(b) Let $G = SU(2)$, and let ρ_V be a continuous representation of G on a finite-dimensional vector space V.

(i) Define the character χ_V of ρ_V, and show that $\chi_V \in \mathbb{N}[z, z^{-1}]$.

(ii) Show that $\chi_V(z) = \chi_V(z^{-1})$.

(iii) Let V be the irreducible 4-dimensional representation of G. Decompose $V \otimes V$ into irreducible representations. Hence decompose the exterior square $\Lambda^2 V$ into irreducible representations.

16H Number Fields

Let K be a number field. State Dirichlet's unit theorem, defining all the terms you use, and what it implies for a quadratic field $\mathbb{Q}(\sqrt{d})$, where $d \neq 0, 1$ is a square-free integer.

Find a fundamental unit of $\mathbb{Q}(\sqrt{26})$.

Find all integral solutions (x, y) of the equation $x^2 - 26y^2 = \pm 10$.

17F Galois Theory

(i) Prove that a finite solvable extension $K \subseteq L$ of fields of characteristic zero is a radical extension.

(ii) Let $x_1, \ldots, x_7$ be variables, $L = \mathbb{Q}(x_1, \ldots, x_7)$, and $K = \mathbb{Q}(e_1, \ldots, e_7)$ where e_i are the elementary symmetric polynomials in the variables x_i. Is there an element $\alpha \in L$ such that $\alpha^2 \in K$ but $\alpha \notin K$? Justify your answer.

(iii) Find an example of a field extension $K \subseteq L$ of degree two such that $L \neq K(\sqrt{\alpha})$ for any $\alpha \in K$. Give an example of a field which has no extension containing an 11th primitive root of unity.

18H Algebraic Topology

State the Mayer–Vietoris theorem for a simplicial complex K which is the union of two subcomplexes M and N. Explain briefly how the connecting homomorphism $\partial_n : H_n(K) \to H_{n-1}(M \cap N)$ is defined.

If K is the union of subcomplexes $M_1, M_2, \ldots, M_n$, with $n \geqslant 2$, such that each intersection

$$M_{i_1} \cap M_{i_2} \cap \cdots \cap M_{i_k}, \qquad 1 \leqslant k \leqslant n,$$

is either empty or has the homology of a point, then show that

$$H_i(K) = 0 \quad \text{for} \quad i \geqslant n - 1.$$

Construct examples for each $n \geqslant 2$ showing that this is sharp.

19G Linear Analysis

Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Define what is meant by an *adjoint* of T and prove that it exists, it is linear and bounded, and that it is unique. [You may use the Riesz Representation Theorem without proof.]

What does it mean to say that T is a *normal operator*? Give an example of a bounded linear map on ℓ_2 that is not normal.

Show that T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.

Prove that if T is normal, then $\sigma(T) = \sigma_{\mathrm{ap}}(T)$, that is, that every element of the spectrum of T is an approximate eigenvalue of T.

20F Algebraic Geometry

(i) Explain how a linear system on a curve C may induce a morphism from C to projective space. What condition on the linear system is necessary to yield a morphism $f : C \to \mathbb{P}^n$ such that the pull-back of a hyperplane section is an element of the linear system? What condition is necessary to imply the morphism is an embedding?

(ii) State the Riemann–Roch theorem for curves.

(iii) Show that any divisor of degree 5 on a curve C of genus 2 induces an embedding.

21G Differential Geometry

Let $\mathrm{U}(n)$ denote the set of $n \times n$ unitary complex matrices. Show that $\mathrm{U}(n)$ is a smooth (real) manifold, and find its dimension. [You may use any general results from the course provided they are stated correctly.] For A any matrix in $\mathrm{U}(n)$ and H an $n \times n$ complex matrix, determine when H represents a tangent vector to $\mathrm{U}(n)$ at A.

Consider the tangent spaces to $\mathrm{U}(n)$ equipped with the metric induced from the standard (Euclidean) inner product $\langle \cdot, \cdot \rangle$ on the real vector space of $n \times n$ complex matrices, given by $\langle L, K \rangle = \mathrm{Re}\ \mathrm{trace}\,(LK^*)$, where Re denotes the real part and K^* denotes the conjugate transpose of K. Suppose that H represents a tangent vector to $\mathrm{U}(n)$ at the identity matrix I. Sketch an explicit construction of a *geodesic curve* on $\mathrm{U}(n)$ passing through I and with tangent direction H, giving a brief proof that the acceleration of the curve is always orthogonal to the tangent space to $\mathrm{U}(n)$.

[*Hint: You will find it easier to work directly with $n \times n$ complex matrices, rather than the corresponding $2n \times 2n$ real matrices.*]

22J Probability and Measure

(a) State Fatou's lemma.

(b) Let X be a random variable on $\mathbb{R}^d$ and let $(X_k)_{k=1}^{\infty}$ be a sequence of random variables on $\mathbb{R}^d$. What does it mean to say that $X_k \to X$ *weakly*?

State and prove the Central Limit Theorem for i.i.d. real-valued random variables. [You may use auxiliary theorems proved in the course provided these are clearly stated.]

(c) Let X be a real-valued random variable with characteristic function φ. Let $(h_n)_{n=1}^{\infty}$ be a sequence of real numbers with $h_n \neq 0$ and $h_n \to 0$. Prove that if we have

$$\liminf_{n\to\infty} \frac{2\varphi(0) - \varphi(-h_n) - \varphi(h_n)}{h_n^2} < \infty,$$

then $\mathbb{E}[X^2] < \infty$.

23K Applied Probability

(i) Let X be a Markov chain on S and $A \subset S$. Let T_A be the hitting time of A and τ_y denote the total time spent at $y \in S$ by the chain before hitting A. Show that if $h(x) = \mathbb{P}_x(T_A < \infty)$, then $\mathbb{E}_x[\tau_y \mid T_A < \infty] = [h(y)/h(x)]\mathbb{E}_x(\tau_y)$.

(ii) Define the Moran model and show that if X_t is the number of individuals carrying allele a at time $t \geqslant 0$ and τ is the fixation time of allele a, then

$$\mathbb{P}(X_\tau = N \mid X_0 = i) = \frac{i}{N}.$$

Show that conditionally on fixation of an allele a being present initially in i individuals,

$$\mathbb{E}[\tau \mid \text{fixation}] = N - i + \frac{N-i}{i} \sum_{j=1}^{i-1} \frac{j}{N-j}.$$

24J Principles of Statistics

Given independent and identically distributed observations $X_1, \ldots, X_n$ with finite mean $E(X_1) = \mu$ and variance $\mathrm{Var}(X_1) = \sigma^2$, explain the notion of a *bootstrap sample* $X_1^b, \ldots, X_n^b$, and discuss how you can use it to construct a confidence interval C_n for μ.

Suppose you can operate a random number generator that can simulate independent uniform random variables $U_1, \ldots, U_n$ on $[0,1]$. How can you use such a random number generator to simulate a bootstrap sample?

Suppose that $(F_n : n \in \mathbb{N})$ and F are cumulative probability distribution functions defined on the real line, that $F_n(t) \to F(t)$ as $n \to \infty$ for every $t \in \mathbb{R}$, and that F is continuous on $\mathbb{R}$. Show that, as $n \to \infty$,

$$\sup_{t \in \mathbb{R}} |F_n(t) - F(t)| \to 0.$$

State (without proof) the theorem about the consistency of the bootstrap of the mean, and use it to give an asymptotic justification of the confidence interval C_n. That is, prove that as $n \to \infty$, $P^{\mathbb{N}}(\mu \in C_n) \to 1 - \alpha$ where $P^{\mathbb{N}}$ is the joint distribution of $X_1, X_2, \ldots$.

[You may use standard facts of stochastic convergence and the Central Limit Theorem without proof.]

25K Optimization and Control

Consider the scalar system evolving as

$$x_t = x_{t-1} + u_{t-1} + \epsilon_t, \quad t = 1, 2, \ldots,$$

where $\{\epsilon_t\}_{t=1}^{\infty}$ is a white noise sequence with $E\epsilon_t = 0$ and $E\epsilon_t^2 = v$. It is desired to choose controls $\{u_t\}_{t=0}^{h-1}$ to minimize $E\left[\sum_{t=0}^{h-1}\left(\frac{1}{2}x_t^2 + u_t^2\right) + x_h^2\right]$. Show that for $h = 6$ the minimal cost is $x_0^2 + 6v$.

Find a constant λ and a function ϕ which solve

$$\phi(x) + \lambda = \min_u \left[\tfrac{1}{2}x^2 + u^2 + E\phi(x + u + \epsilon_1)\right].$$

Let P be the class of those policies for which every u_t obeys the constraint $(x_t + u_t)^2 \leqslant (0.9)x_t^2$. Show that $E_\pi \phi(x_t) \leqslant x_0^2 + 10v$, for all $\pi \in P$. Find, and prove optimal, a policy which over all $\pi \in P$ minimizes

$$\lim_{h \to \infty} \frac{1}{h} E_\pi \left[\sum_{t=0}^{h-1}\left(\tfrac{1}{2}x_t^2 + u_t^2\right)\right].$$

26K Stochastic Financial Models

(i) An investor in a single-period market with time-0 wealth w_0 may generate any time-1 wealth w_1 of the form $w_1 = w_0 + X$, where X is any element of a vector space V of random variables. The investor's objective is to maximize $E[U(w_1)]$, where U is strictly increasing, concave and C^2. Define the *utility indifference price* $\pi(Y)$ of a random variable Y.

Prove that the map $Y \mapsto \pi(Y)$ is concave. [You may assume that any supremum is attained.]

(ii) Agent j has utility $U_j(x) = -\exp(-\gamma_j x)$, $j = 1, \ldots, J$. The agents may buy for time-0 price p a risky asset which will be worth X at time 1, where X is random and has density

$$f(x) = \tfrac{1}{2}\alpha e^{-\alpha|x|}, \qquad -\infty < x < \infty .$$

Assuming zero interest, prove that agent j will optimally choose to buy

$$\theta_j = -\frac{\sqrt{1+p^2\alpha^2} - 1}{\gamma_j \, p}$$

units of the risky asset at time 0.

If the asset is in unit net supply, if $\Gamma^{-1} \equiv \sum_j \gamma_j^{-1}$, and if $\alpha > \Gamma$, prove that the market for the risky asset will clear at price

$$p = -\frac{2\Gamma}{\alpha^2 - \Gamma^2} .$$

What happens if $\alpha \leqslant \Gamma$?

27C Asymptotic Methods

Consider the ordinary differential equation

$$\frac{d^2u}{dz^2} + f(z)\frac{du}{dz} + g(z)u = 0\,,$$

where

$$f(z) \sim \sum_{m=0}^{\infty} \frac{f_m}{z^m}\,, \qquad g(z) \sim \sum_{m=0}^{\infty} \frac{g_m}{z^m}\,, \qquad z \to \infty\,,$$

and f_m, g_m are constants. Look for solutions in the asymptotic form

$$u(z) = e^{\lambda z} z^{\mu} \left[1 + \frac{a}{z} + \frac{b}{z^2} + O\left(\frac{1}{z^3}\right)\right]\,, \qquad z \to \infty\,,$$

and determine λ in terms of (f_0, g_0), as well as μ in terms of (λ, f_0, f_1, g_1).

Deduce that the Bessel equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)u = 0\,,$$

where ν is a complex constant, has two solutions of the form

$$u^{(1)}(z) = \frac{e^{iz}}{z^{1/2}} \left[1 + \frac{a^{(1)}}{z} + O\left(\frac{1}{z^2}\right)\right]\,, \quad z \to \infty\,,$$

$$u^{(2)}(z) = \frac{e^{-iz}}{z^{1/2}} \left[1 + \frac{a^{(2)}}{z} + O\left(\frac{1}{z^2}\right)\right]\,, \quad z \to \infty\,,$$

and determine $a^{(1)}$ and $a^{(2)}$ in terms of ν.

Can the above asymptotic expansions be valid for all $\arg(z)$, or are they valid only in certain domains of the complex z-plane? Justify your answer briefly.

28B Dynamical Systems

Let $f : I \to I$ be a continuous one-dimensional map of an interval $I \subset \mathbb{R}$. Explain what is meant by the statements (i) that f has a *horseshoe* and (ii) that f is *chaotic* (according to Glendinning's definition).

Assume that f has a 3-cycle $\{x_0, x_1, x_2\}$ with $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_0 = f(x_2)$ and, without loss of generality, $x_0 < x_1 < x_2$. Prove that f^2 has a horseshoe. [You may assume the intermediate value theorem.]

Represent the effect of f on the intervals $I_a = [x_0, x_1]$ and $I_b = [x_1, x_2]$ by means of a directed graph, explaining carefully how the graph is constructed. Explain what feature of the graph implies the existence of a 3-cycle.

The map $g : I \to I$ has a 5-cycle $\{x_0, x_1, x_2, x_3, x_4\}$ with $x_{i+1} = g(x_i)$, $0 \leqslant i \leqslant 3$ and $x_0 = g(x_4)$, and $x_0 < x_1 < x_2 < x_3 < x_4$. For which n, $1 \leqslant n \leqslant 4$, is an n-cycle of g guaranteed to exist? Is g guaranteed to be chaotic? Is g guaranteed to have a horseshoe? Justify your answers. [You may use a suitable directed graph as part of your arguments.]

How do your answers to the above change if instead $x_4 < x_2 < x_1 < x_3 < x_0$?

29E Partial Differential Equations

(a) Show that the Cauchy problem for $u(x, t)$ satisfying

$$u_t + u = u_{xx}$$

with initial data $u(x, 0) = u_0(x)$, which is a smooth 2π-periodic function of x, defines a *strongly continuous one parameter semi-group of contractions* on the Sobolev space H^s_{per} for any $s \in \{0, 1, 2, \dots\}$.

(b) Solve the Cauchy problem for the equation

$$u_{tt} + u_t + \frac{1}{4}u = u_{xx}$$

with $u(x, 0) = u_0(x)\,,\ u_t(x, 0) = u_1(x)$, where u_0, u_1 are smooth 2π-periodic functions of x, and show that the solution is smooth. Prove from first principles that the solution satisfies the property of *finite propagation speed.*

[In this question all functions are real-valued, and

$$H^s_{\text{per}} = \Big\{ u = \sum_{m \in \mathbb{Z}} \hat{u}(m) e^{imx} \in L^2 \,:\, \|u\|^2_{H^s} = \sum_{m \in \mathbb{Z}} (1 + m^2)^s |\hat{u}(m)|^2 < \infty \Big\}$$

are the Sobolev spaces of functions which are 2π-periodic in x, for $s = 0, 1, 2, \dots$.]

30A Principles of Quantum Mechanics

The Hamiltonian for a quantum system in the Schrödinger picture is $H_0 + \lambda V(t)$, where H_0 is independent of time and the parameter λ is small. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Suppose that $|\chi\rangle$ and $|\phi\rangle$ are eigenstates of H_0 with distinct eigenvalues E and E', respectively. Show that if the system is in state $|\chi\rangle$ at time zero then the probability of measuring it to be in state $|\phi\rangle$ at time t is

$$\frac{\lambda^2}{\hbar^2}\left|\int_0^t dt'\,\langle\phi|V(t')|\chi\rangle\, e^{i(E'-E)t'/\hbar}\right|^2 \;+\; O(\lambda^3)\,.$$

Let H_0 be the Hamiltonian for an isotropic three-dimensional harmonic oscillator of mass m and frequency ω, with $\chi(r)$ being the ground state wavefunction (where $r = |\mathbf{x}|$) and $\phi_i(\mathbf{x}) = (2m\omega/\hbar)^{1/2}\, x_i\, \chi(r)$ being wavefunctions for the states at the first excited energy level ($i = 1, 2, 3$). The oscillator is in its ground state at $t = 0$ when a perturbation

$$\lambda V(t) \;=\; \lambda\, \hat{x}_3\, e^{-\mu t}$$

is applied, with $\mu > 0$, and H_0 is then measured after a very large time has elapsed. Show that to first order in perturbation theory the oscillator will be found in one particular state at the first excited energy level with probability

$$\frac{\lambda^2}{2\hbar m\omega\,(\mu^2+\omega^2)}\,,$$

but that the probability that it will be found in either of the other excited states is zero (to this order).

$\Big[$ You may use the fact that $4\pi\displaystyle\int_0^\infty r^4\,|\chi(r)|^2\,dr = \frac{3\hbar}{2m\omega}\,.$ $\Big]$

31A Applications of Quantum Mechanics

Let Λ be a Bravais lattice with basis vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Define the reciprocal lattice Λ^* and write down basis vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ for Λ^* in terms of the basis for Λ.

A finite crystal consists of identical atoms at sites of Λ given by

$$\boldsymbol{\ell} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3 \qquad \text{with} \qquad 0 \leqslant n_i < N_i \; .$$

A particle of mass m scatters off the crystal; its wavevector is $\mathbf{k}$ before scattering and $\mathbf{k}'$ after scattering, with $|\mathbf{k}| = |\mathbf{k}'|$. Show that the scattering amplitude in the Born approximation has the form

$$-\frac{m}{2\pi\hbar^2}\,\Delta(\mathbf{q})\,\tilde{U}(\mathbf{q}) \; , \qquad \mathbf{q} = \mathbf{k}' - \mathbf{k} \; ,$$

where $U(\mathbf{x})$ is the potential due to a single atom at the origin and $\Delta(\mathbf{q})$ depends on the crystal structure. [You may assume that in the Born approximation the amplitude for scattering off a potential $V(\mathbf{x})$ is $-(m/2\pi\hbar^2)\,\tilde{V}(\mathbf{q})$ where tilde denotes the Fourier transform.]

Derive an expression for $|\,\Delta(\mathbf{q})\,|$ that is valid when $e^{-i\mathbf{q}\cdot\mathbf{a}_i} \neq 1$. Show also that when $\mathbf{q}$ is a reciprocal lattice vector $|\,\Delta(\mathbf{q})\,|$ is equal to the total number of atoms in the crystal. Comment briefly on the significance of these results.

Now suppose that Λ is a face-centred-cubic lattice:

$$\mathbf{a}_1 = \frac{a}{2}(\hat{\mathbf{y}} + \hat{\mathbf{z}}) \; , \quad \mathbf{a}_2 = \frac{a}{2}(\hat{\mathbf{z}} + \hat{\mathbf{x}}) \; , \quad \mathbf{a}_3 = \frac{a}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}})$$

where a is a constant. Show that for a particle incident with $|\mathbf{k}| > 2\pi/a$, enhanced scattering is possible for at least two values of the scattering angle, θ_1 and θ_2, related by

$$\frac{\sin(\theta_1/2)}{\sin(\theta_2/2)} = \frac{\sqrt{3}}{2} \; .$$

32C Statistical Physics

The Ising model consists of N particles, labelled by i, arranged on a D-dimensional Euclidean lattice with periodic boundary conditions. Each particle has spin *up* $s_i = +1$, or *down* $s_i = -1$, and the energy in the presence of a magnetic field B is

$$E = -B \sum_i s_i - J \sum_{\langle i,j \rangle} s_i \, s_j \, ,$$

where $J > 0$ is a constant and $\langle i,j \rangle$ indicates that the second sum is over each pair of nearest neighbours (every particle has $2D$ nearest neighbours). Let $\beta = 1/k_B T$, where T is the temperature.

(i) Express the average spin per particle, $m = (\sum_i \langle s_i \rangle)/N$, in terms of the canonical partition function Z.

(ii) Show that in the mean-field approximation

$$Z = C \, [\, Z_1(\beta B_{\text{eff}}) \,]^N$$

where Z_1 is a single-particle partition function, B_{eff} is an effective magnetic field which you should find in terms of B, J, D and m, and C is a prefactor which you should also evaluate.

(iii) Deduce an equation that determines m for general values of B, J and temperature T. Without attempting to solve for m explicitly, discuss how the behaviour of the system depends on temperature when $B = 0$, deriving an expression for the critical temperature T_c and explaining its significance.

(iv) Comment briefly on whether the results obtained using the mean-field approximation for $B = 0$ are consistent with an expression for the free energy of the form

$$F(m,T) \; = \; F_0(T) + \frac{a}{2}(T - T_c)m^2 + \frac{b}{4}m^4$$

where a and b are positive constants.

33A Electrodynamics

A point particle of charge q has trajectory $y^\mu(\tau)$ in Minkowski space, where τ is its proper time. The resulting electromagnetic field is given by the Liénard–Wiechert 4-potential

$$A^\mu(x) \;=\; -\frac{q\,\mu_0\,c}{4\pi}\,\frac{u^\mu(\tau_*)}{R^\nu(\tau_*)\,u_\nu(\tau_*)}\,, \quad \text{where} \quad R^\nu = x^\nu - y^\nu(\tau) \quad \text{and} \quad u^\mu = dy^\mu/d\tau\,.$$

Write down the condition that determines the point $y^\mu(\tau_*)$ on the trajectory of the particle for a given value of x^μ. Express this condition in terms of components, setting $x^\mu = (ct, \mathbf{x})$ and $y^\mu = (ct', \mathbf{y})$, and define the retarded time t_r.

Suppose that the 3-velocity of the particle $\mathbf{v}(t') = \dot{\mathbf{y}}(t') = d\mathbf{y}/dt'$ is small in size compared to c, and suppose also that $r = |\mathbf{x}| \gg |\mathbf{y}|$. Working to leading order in $1/r$ and to first order in $\mathbf{v}$, show that

$$\phi(x) \;=\; \frac{q\,\mu_0\,c}{4\pi r}\,(\,c + \hat{\mathbf{r}}\cdot\mathbf{v}(t_r)\,)\,, \quad \mathbf{A}(x) \;=\; \frac{q\,\mu_0}{4\pi r}\,\mathbf{v}(t_r)\,, \quad \text{where} \quad \hat{\mathbf{r}} = \mathbf{x}/r\,.$$

Now assume that t_r can be replaced by $t_- = t - (r/c)$ in the expressions for ϕ and $\mathbf{A}$ above. Calculate the electric and magnetic fields to leading order in $1/r$ and hence show that the Poynting vector is (in this approximation)

$$\mathbf{N}(x) \;=\; \frac{q^2\mu_0}{(4\pi)^2\,c}\,\frac{\hat{\mathbf{r}}}{r^2}\left|\,\hat{\mathbf{r}}\times\dot{\mathbf{v}}(t_-)\,\right|^2.$$

If the charge q is performing simple harmonic motion $\mathbf{y}(t') = A\mathbf{n}\cos\omega t'$, where $\mathbf{n}$ is a unit vector and $A\omega \ll c$, find the total energy radiated during one period of oscillation.

34D General Relativity

In static spherically symmetric coordinates, the metric g_{ab} for de Sitter space is given by

$$ds^2 = -(1 - r^2/a^2)dt^2 + (1 - r^2/a^2)^{-1}dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and a is a constant.

(a) Let $u = t - a \tanh^{-1}(r/a)$ for $r \leqslant a$. Use the (u, r, θ, ϕ) coordinates to show that the surface $r = a$ is non-singular. Is $r = 0$ a space-time singularity?

(b) Show that the vector field $g^{ab}u_{,a}$ is null.

(c) Show that the radial null geodesics must obey either

$$\frac{du}{dr} = 0 \qquad \text{or} \qquad \frac{du}{dr} = -\frac{2}{1 - r^2/a^2}\,.$$

Which of these families of geodesics is outgoing $(dr/dt > 0)$?

Sketch these geodesics in the (u, r) plane for $0 \leqslant r \leqslant a$, where the r-axis is horizontal and lines of constant u are inclined at $45°$ to the horizontal.

(d) Show, by giving an explicit example, that an observer moving on a timelike geodesic starting at $r = 0$ can cross the surface $r = a$ within a finite proper time.

35E Fluid Dynamics II

A stationary inviscid fluid of thickness h and density ρ is located below a free surface at $y = h$ and above a deep layer of inviscid fluid of the same density in $y < 0$ flowing with uniform velocity $U > 0$ in the $\mathbf{e}_x$ direction. The base velocity profile is thus

$$u = U,\ y < 0; \quad u = 0,\ 0 < y < h,$$

while the free surface at $y = h$ is maintained flat by gravity.

By considering small perturbations of the vortex sheet at $y = 0$ of the form $\eta = \eta_0 e^{ikx+\sigma t}$, $k > 0$, calculate the dispersion relationship between k and σ in the irrotational limit. By explicitly deriving that

$$\mathrm{Re}(\sigma) = \pm\frac{\sqrt{\tanh(hk)}}{1 + \tanh(hk)}Uk,$$

deduce that the vortex sheet is unstable at all wavelengths. Show that the growth rates of the unstable modes are approximately $Uk/2$ when $hk \gg 1$ and $Uk\sqrt{hk}$ when $hk \ll 1$.

36B Waves

The shallow-water equations

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

describe one-dimensional flow over a horizontal boundary with depth $h(x,t)$ and velocity $u(x,t)$, where g is the acceleration due to gravity.

Show that the Riemann invariants $u \pm 2(c - c_0)$ are constant along characteristics $C_\pm$ satisfying $dx/dt = u \pm c$, where $c(h)$ is the linear wave speed and c_0 denotes a reference state.

An initially stationary pool of fluid of depth h_0 is held between a stationary wall at $x = a > 0$ and a removable barrier at $x = 0$. At $t = 0$ the barrier is instantaneously removed allowing the fluid to flow into the region $x < 0$.

For $0 \leqslant t \leqslant a/c_0$, find $u(x,t)$ and $c(x,t)$ in each of the regions

$$\begin{array}{ll} \text{(i)} & c_0 t \leqslant x \leqslant a \\ \text{(ii)} & -2c_0 t \leqslant x \leqslant c_0 t \end{array}$$

explaining your argument carefully with a sketch of the characteristics in the (x,t) plane.

For $t \geqslant a/c_0$, show that the solution in region (ii) above continues to hold in the region $-2c_0 t \leqslant x \leqslant 3a(c_0 t/a)^{1/3} - 2c_0 t$. Explain why this solution does not hold in $3a(c_0 t/a)^{1/3} - 2c_0 t < x < a$.

37E Numerical Analysis

(a) Define the mth *Krylov space* $K_m(A, v)$ for $A \in \mathbb{R}^{n \times n}$ and $0 \neq v \in \mathbb{R}^n$. Letting δ_m be the dimension of $K_m(A, v)$, prove the following results.

(i) There exists a positive integer $s \leqslant n$ such that $\delta_m = m$ for $m \leqslant s$ and $\delta_m = s$ for $m > s$.

(ii) If $v = \sum_{i=1}^{s'} c_i w_i$, where w_i are eigenvectors of A for distinct eigenvalues and all c_i are nonzero, then $s = s'$.

(b) Define the term *residual* in the conjugate gradient (CG) method for solving a system $Ax = b$ with symmetric positive definite A. Explain (without proof) the connection to Krylov spaces and prove that for any right-hand side b the CG method finds an exact solution after at most t steps, where t is the number of distinct eigenvalues of A. [You may use without proof known properties of the iterates of the CG method.]

Define what is meant by preconditioning, and explain two ways in which preconditioning can speed up convergence. Can we choose the preconditioner so that the CG method requires only one step? If yes, is it a reasonable method for speeding up the computation?

END OF PAPER