UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Tuesday, 2 June, 2015 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *. . . ,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Define the Euler totient function ϕ and the Möbius function μ. Suppose f and g are functions defined on the natural numbers satisfying $f(n) = \sum_{d|n} g(d)$. State and prove a formula for g in terms of f. Find a relationship between μ and ϕ.

Define the Riemann zeta function $\zeta(s)$. Find a Dirichlet series for $\zeta(s-1)/\zeta(s)$ valid for $\mathrm{Re}(s) > 2$.

2I Topics in Analysis

Let $x_1, x_2, \ldots, x_n$ be the roots of the Legendre polynomial of degree n. Let $A_1, A_2, \ldots, A_n$ be chosen so that

$$\int_{-1}^{1} p(t)\,dt = \sum_{j=1}^{n} A_j p(x_j)$$

for all polynomials p of degree $n-1$ or less. Assuming any results about Legendre polynomials that you need, prove the following results:

(i) $\displaystyle\int_{-1}^{1} p(t)\,dt = \sum_{j=1}^{n} A_j p(x_j)$ for all polynomials p of degree $2n-1$ or less;

(ii) $A_j \geqslant 0$ for all $1 \leqslant j \leqslant n$;

(iii) $\displaystyle\sum_{j=1}^{n} A_j = 2$.

Now consider $Q_n(f) = \sum_{j=1}^{n} A_j f(x_j)$. Show that

$$Q_n(f) \to \int_{-1}^{1} f(t)\,dt$$

as $n \to \infty$ for all continuous functions f.

3G Coding and Cryptography

A random variable A takes values in the alphabet $\mathcal{A} = \{a, b, c, d, e\}$ with probabilities 0.4, 0.2, 0.2, 0.1 and 0.1. Calculate the entropy of A.

Define what it means for a code for a general finite alphabet to be *optimal*. Find such a code for the distribution above and show that there are optimal codes for this distribution with differing lengths of codeword.

[You may use any results from the course without proof. Note that $\log_2 5 \simeq 2.32$.]

4J Statistical Modelling

Let $Y_1, \ldots, Y_n$ be independent Poisson random variables with means $\mu_1, \ldots, \mu_n$, where $\log(\mu_i) = \beta x_i$ for some known constants $x_i \in \mathbb{R}$ and an unknown parameter β. Find the log-likelihood for β.

By first computing the first and second derivatives of the log-likelihood for β, describe the algorithm you would use to find the maximum likelihood estimator $\hat{\beta}$. [*Hint: Recall that if $Z \sim Pois(\mu)$ then*

$$\mathbb{P}(Z = k) = \frac{\mu^k e^{-\mu}}{k!}$$

for $k \in \{0, 1, 2, \ldots\}$.]

5E Mathematical Biology

An activator-inhibitor system is described by the equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= 2u + u^2 - u\,v + \frac{\partial^2 u}{\partial x^2}\,, \\ \frac{\partial v}{\partial t} &= a\left(u^2 - v\right) + d\,\frac{\partial^2 v}{\partial x^2}\,, \end{aligned}$$

where $a, d > 0$.

Find the range of a for which the spatially homogeneous system has a stable equilibrium solution with $u > 0$ and $v > 0$.

For the case when the homogeneous system is stable, consider spatial perturbations proportional to $\cos(kx)$ to the equilibrium solution found above. Show that the system has a Turing instability when

$$d > \left(\tfrac{7}{2} + 2\sqrt{3}\right) a\,.$$

6B Further Complex Methods

Give a brief description of what is meant by *analytic continuation*.

The dilogarithm function is defined by

$$\mathrm{Li}_2(z) \;=\; \sum_{n=1}^{\infty} \frac{z^n}{n^2}\,, \qquad |z| < 1\,.$$

Let

$$f(z) \;=\; -\int_C \frac{1}{u}\ln(1-u)\,du$$

where C is a contour that runs from the origin to the point z. Show that $f(z)$ provides an analytic continuation of $\mathrm{Li}_2(z)$ and describe its domain of definition in the complex plane, given a suitable branch cut.

7D Classical Dynamics

The Lagrangian for a heavy symmetric top of mass M, pinned at a point that is a distance l from the centre of mass, is

$$L = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta\,.$$

(a) Find all conserved quantities. In particular, show that ω_3, the spin of the top, is constant.

(b) Show that θ obeys the equation of motion

$$I_1\ddot{\theta} = -\frac{dV_{\text{eff}}}{d\theta}\,,$$

where the explicit form of V_{eff} should be determined.

(c) Determine the condition for uniform precession with no nutation, that is $\dot{\theta} = 0$ and $\dot{\phi} = \text{const}$. For what values of ω_3 does such uniform precession occur?

8C Cosmology

The mass density perturbation equation for non-relativistic matter ($P \ll \rho c^2$) with wave number k in the late universe ($t > t_{\rm eq}$) is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \left(4\pi G\rho - \frac{c_s^2\, k^2}{a^2}\right)\delta = 0\,. \qquad (*)$$

Suppose that a non-relativistic fluid with the equation of state $P \propto \rho^{4/3}$ dominates the universe when $a(t) = t^{2/3}$, and the curvature and the cosmological constant can be neglected. Show that the sound speed can be written in the form $c_s^2(t) \equiv dP/d\rho = \bar{c}_s^2\, t^{-2/3}$ where $\bar{c}_s$ is a constant.

Find power-law solutions to $(*)$ of the form $\delta \propto t^\beta$ and hence show that the general solution is

$$\delta = A_k\, t^{n_+} + B_k\, t^{n_-}$$

where

$$n_\pm = -\frac{1}{6} \pm \left[\left(\frac{5}{6}\right)^2 - \bar{c}_s^2\, k^2\right]^{1/2}.$$

Interpret your solutions in the two regimes $k \ll k_J$ and $k \gg k_J$ where $k_J = \dfrac{5}{6\bar{c}_s}$.

SECTION II

9I Topics in Analysis

State and prove Sperner's lemma concerning the colouring of triangles.

Deduce a theorem, to be stated clearly, on retractions to the boundary of a disc.

State Brouwer's fixed point theorem for a disc and sketch a proof of it.

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous function such that for some $K > 0$ we have $\|g(x) - x\| \leqslant K$ for all $x \in \mathbb{R}^2$. Show that g is surjective.

10G Coding and Cryptography

Briefly describe the *RSA public key cipher*.

Just before it went into liquidation, the Internet Bank decided that it wanted to communicate with each of its customers using an RSA cipher. So, it chose a large modulus N, which is the product of two large prime numbers, and chose encrypting exponents e_j and decrypting exponents d_j for each customer j. The bank published N and e_j and sent the decrypting exponent d_j secretly to customer j. Show explicitly that the cipher can be broken by each customer.

The bank sent out the same message to each customer. I am not a customer of the bank but have two friends who are and I notice that their published encrypting exponents are coprime. Explain how I can find the original message. Can I break the cipher?

11B Further Complex Methods

The Riemann zeta function is defined by the sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which converges for $\operatorname{Re} s > 1$. Show that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt\,, \quad \operatorname{Re} s > 1\,. \qquad (*)$$

The analytic continuation of $\zeta(s)$ is given by the Hankel contour integral

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{0+} \frac{t^{s-1}}{e^{-t} - 1}\, dt\,.$$

Verify that this agrees with the integral $(*)$ above when $\operatorname{Re} s > 1$ and s is not an integer. [You may assume $\Gamma(s)\Gamma(1-s) = \pi / \sin \pi s$.] What happens when $s = 2, 3, 4, \ldots$?

Evaluate $\zeta(0)$. Show that $(e^{-t} - 1)^{-1} + \frac{1}{2}$ is an odd function of t and hence, or otherwise, show that $\zeta(-2n) = 0$ for any positive integer n.

12C Classical Dynamics

(a) Consider a Lagrangian dynamical system with one degree of freedom. Write down the expression for the Hamiltonian of the system in terms of the generalized velocity $\dot{q}$, momentum p, and the Lagrangian $L(q, \dot{q}, t)$. By considering the differential of the Hamiltonian, or otherwise, derive Hamilton's equations.

Show that if q is ignorable (cyclic) with respect to the Lagrangian, i.e. $\partial L / \partial q = 0$, then it is also ignorable with respect to the Hamiltonian.

(b) A particle of charge q and mass m moves in the presence of electric and magnetic fields such that the scalar and vector potentials are $\phi = yE$ and $\mathbf{A} = (0, xB, 0)$, where (x, y, z) are Cartesian coordinates and E, B are constants. The Lagrangian of the particle is

$$L = \frac{1}{2} m \dot{\mathbf{r}}^2 - q\phi + q\dot{\mathbf{r}} \cdot \mathbf{A}\,.$$

Starting with the Lagrangian, derive an explicit expression for the Hamiltonian and use Hamilton's equations to determine the motion of the particle.

13I Logic and Set Theory

(a) Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent. Give the inductive definitions of ordinal multiplication and ordinal exponentiation.

(b) Answer, with brief justification, the following:

(i) For ordinals α, β and γ with $\alpha < \beta$, must we have $\alpha + \gamma < \beta + \gamma$? Must we have $\gamma + \alpha < \gamma + \beta$?

(ii) For ordinals α and β with $\alpha < \beta$, must we have $\alpha^{\omega} < \beta^{\omega}$?

(iii) Is there an ordinal $\alpha > 1$ such that $\alpha^{\omega} = \alpha$?

(iv) Show that $\omega^{\omega_1} = \omega_1$. Is ω_1 the least ordinal α such that $\omega^{\alpha} = \alpha$?

[You may use standard facts about ordinal arithmetic.]

14I Graph Theory

(a) Define the *Ramsey numbers* $R(s,t)$ and $R(s)$ for integers s, $t \geqslant 2$. Show that $R(s,t)$ exists for all s, $t \geqslant 2$ and that if s, $t \geqslant 3$ then $R(s,t) \leqslant R(s-1,t) + R(s,t-1)$.

(b) Show that, as $s \to \infty$, we have $R(s) = O(4^s)$ and $R(s) = \Omega(2^{s/2})$.

(c) Show that, as $t \to \infty$, we have $R(3,t) = O(t^2)$ and $R(3,t) = \Omega\left(\left(\frac{t}{\log t}\right)^{3/2}\right)$.

[*Hint: For the lower bound in* (c), *you may wish to begin by modifying a random graph to show that for all n and p we have*

$$R(3,t) > n - \binom{n}{3}p^3 - \binom{n}{t}(1-p)^{\binom{t}{2}}. \quad]$$

15F Representation Theory

Let G be a finite group. Suppose that $\rho : G \to \mathrm{GL}(V)$ is a finite-dimensional complex representation of dimension d. Let $n \in \mathbb{N}$ be arbitrary.

(i) Define the nth *symmetric power* S^nV and the nth *exterior power* Λ^nV and write down their respective dimensions.

Let $g \in G$ and let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of g on V. What are the eigenvalues of g on S^nV and on Λ^nV?

(ii) Let X be an indeterminate. For any $g \in G$, define the *characteristic polynomial* $Q = Q(g, X)$ of g on V by $Q(g, X) := \det(g - XI)$. What is the relationship between the coefficients of Q and the character χ_{Λ^nV} of the exterior power?

Find a relation between the character χ_{S^nV} of the symmetric power and the polynomial Q.

16H Number Fields

(i) Let $d \equiv 2$ or $3 \mod 4$. Show that (p) remains prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ if and only if $x^2 - d$ is irreducible mod p.

(ii) Factorise (2), (3) in $\mathcal{O}_K$, when $K = \mathbb{Q}(\sqrt{-14})$. Compute the class group of K.

17F Galois Theory

(i) State the fundamental theorem of Galois theory, without proof. Let L be a splitting field of $t^3 - 2 \in \mathbb{Q}[t]$. Show that $\mathbb{Q} \subseteq L$ is Galois and that $\mathrm{Gal}(L/\mathbb{Q})$ has a subgroup which is not normal.

(ii) Let Φ_8 be the 8th cyclotomic polynomial and denote its image in $\mathbb{F}_7[t]$ again by Φ_8. Show that Φ_8 is not irreducible in $\mathbb{F}_7[t]$.

(iii) Let m and n be coprime natural numbers, and let $\mu_m = \exp(2\pi i/m)$ and $\mu_n = \exp(2\pi i/n)$ where $i = \sqrt{-1}$. Show that $\mathbb{Q}(\mu_m) \cap \mathbb{Q}(\mu_n) = \mathbb{Q}$.

18H Algebraic Topology

Define what it means for $p : \widetilde{X} \to X$ to be a covering map, and what it means to say that p is a universal cover.

Let $p : \widetilde{X} \to X$ be a universal cover, $A \subset X$ be a locally path connected subspace, and $\widetilde{A} \subset p^{-1}(A)$ be a path component containing a point $\tilde{a}_0$ with $p(\tilde{a}_0) = a_0$. Show that the restriction $p|_{\widetilde{A}} : \widetilde{A} \to A$ is a covering map, and that under the Galois correspondence it corresponds to the subgroup

$$\mathrm{Ker}\big(\, \pi_1(A, a_0) \to \pi_1(X, a_0)\,\big)$$

of $\pi_1(A, a_0)$.

19G Linear Analysis

(a) Let $T\colon X \to Y$ be a linear map between normed spaces. What does it mean to say that T is *bounded*? Show that T is bounded if and only if T is continuous. Define the *operator norm* of T and show that the set $\mathcal{B}(X, Y)$ of all bounded, linear maps from X to Y is a normed space in the operator norm.

(b) For each of the following linear maps T, determine if T is bounded. When T is bounded, compute its operator norm and establish whether T is compact. Justify your answers. Here $\|f\|_\infty = \sup_{t\in[0,1]}|f(t)|$ for $f \in C[0,1]$ and $\|f\| = \|f\|_\infty + \|f'\|_\infty$ for $f \in C^1[0,1]$.

(i) $T\colon \big(C^1[0,1], \|\cdot\|_\infty\big) \to \big(C^1[0,1], \|\cdot\|\big)$, $T(f) = f$.

(ii) $T\colon \big(C^1[0,1], \|\cdot\|\big) \to (C[0,1], \|\cdot\|_\infty)$, $T(f) = f$.

(iii) $T\colon \big(C^1[0,1], \|\cdot\|\big) \to (C[0,1], \|\cdot\|_\infty)$, $T(f) = f'$.

(iv) $T\colon (C[0,1], \|\cdot\|_\infty) \to \mathbb{R}$, $T(f) = \displaystyle\int_0^1 f(t)h(t)\,dt$, where h is a given element of $C[0,1]$. [*Hint: Consider first the case that $h(x) \neq 0$ for every $x \in [0,1]$, and apply T to a suitable function. In the general case apply T to a suitable sequence of functions.*]

20F Riemann Surfaces

Let G be a domain in $\mathbb{C}$. Define the germ of a function element (f, D) at $z \in D$. Let $\mathcal{G}$ be the set of all germs of function elements in G. Define the topology on $\mathcal{G}$. Show it is a topology, and that it is Hausdorff. Define the complex structure on $\mathcal{G}$, and show that there is a natural projection map $\pi : \mathcal{G} \to G$ which is an analytic covering map on each connected component of $\mathcal{G}$.

Given a complete analytic function $\mathcal{F}$ on G, describe how it determines a connected component $\mathcal{G}_{\mathcal{F}}$ of $\mathcal{G}$. [You may assume that a function element (g, E) is an analytic continuation of a function element (f, D) along a path $\gamma : [0,1] \to G$ if and only if there is a lift of γ to $\mathcal{G}$ starting at the germ of (f, D) at $\gamma(0)$ and ending at the germ of (g, E) at $\gamma(1)$.]

In each of the following cases, give an example of a domain G in $\mathbb{C}$ and a complete analytic function $\mathcal{F}$ such that:

(i) $\pi : \mathcal{G}_{\mathcal{F}} \to G$ is regular but not bijective;

(ii) $\pi : \mathcal{G}_{\mathcal{F}} \to G$ is surjective but not regular.

21F Algebraic Geometry

(i) Define the radical of an ideal.

(ii) Assume the following statement: If k is an algebraically closed field and $I \subseteq k[x_1, \ldots, x_n]$ is an ideal, then either $I = (1)$ or $Z(I) \neq \emptyset$. Prove the Hilbert Nullstellensatz, namely that if $I \subseteq k[x_1, \ldots, x_n]$ with k algebraically closed, then

$$I(Z(I)) = \sqrt{I}.$$

(iii) Show that if A is a commutative ring and $I, J \subseteq A$ are ideals, then

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

(iv) Is

$$\sqrt{I + J} = \sqrt{I} + \sqrt{J}\,?$$

Give a proof or a counterexample.

22G Differential Geometry

If U denotes a domain in $\mathbb{R}^2$, what is meant by saying that a smooth map $\phi : U \to \mathbb{R}^3$ is an *immersion*? Define what it means for such an immersion to be *isothermal*. Explain what it means to say that an immersed surface is *minimal*.

Let $\phi(u,v) = (x(u,v), y(u,v), z(u,v))$ be an isothermal immersion. Show that it is minimal if and only if x, y, z are harmonic functions of u, v. [You may use the formula for the mean curvature given in terms of the first and second fundamental forms, namely $H = (eG - 2fF + gE)/(2\{EG - F^2\})$.]

Produce an example of an immersed minimal surface which is not an open subset of a catenoid, helicoid, or a plane. Prove that your example does give an immersed minimal surface in $\mathbb{R}^3$.

23J Probability and Measure

(a) Let $(E, \mathcal{E}, \mu)$ be a measure space, and let $1 \leqslant p < \infty$. What does it mean to say that f belongs to $L^p(E, \mathcal{E}, \mu)$?

(b) State Hölder's inequality.

(c) Consider the measure space of the unit interval endowed with Lebesgue measure. Suppose $f \in L^2(0,1)$ and let $0 < \alpha < 1/2$.

(i) Show that for all $x \in \mathbb{R}$,

$$\int_0^1 |f(y)||x-y|^{-\alpha}\, dy < \infty\,.$$

(ii) For $x \in \mathbb{R}$, define

$$g(x) = \int_0^1 f(y)|x-y|^{-\alpha} dy\,.$$

Show that for $x \in \mathbb{R}$ fixed, the function g satisfies

$$|g(x+h) - g(x)| \leqslant \|f\|_2 \cdot (I(h))^{1/2},$$

where

$$I(h) = \int_0^1 \left(|x+h-y|^{-\alpha} - |x-y|^{-\alpha}\right)^2 dy.$$

(iii) Prove that g is a continuous function. [*Hint: You may find it helpful to split the integral defining $I(h)$ into several parts.*]

24K Applied Probability

(i) Defne a *Poisson process* on $\mathbb{R}_+$ with rate λ. Let N and M be two independent Poisson processes on $\mathbb{R}_+$ of rates λ and μ respectively. Prove that $N+M$ is also a Poisson process and find its rate.

(ii) Let X be a discrete time Markov chain with transition matrix K on the finite state space S. Find the generator of the continuous time Markov chain $Y_t = X_{N_t}$ in terms of K and λ. Show that if π is an invariant distribution for X, then it is also invariant for Y.

Suppose that X has an absorbing state a. If τ_a and T_a are the absorption times for X and Y respectively, write an equation that relates $\mathbb{E}_x[\tau_a]$ and $\mathbb{E}_x[T_a]$, where $x \in S$.

[*Hint: You may want to prove that if* $\xi_1, \xi_2, \ldots$ *are i.i.d. non-negative random variables with* $\mathbb{E}[\xi_1] < \infty$ *and* M *is an independent non-negative random variable, then* $\mathbb{E}\left[\sum_{i=1}^{M} \xi_i\right] = \mathbb{E}[M]\,\mathbb{E}[\xi_1]$.]

25J Principles of Statistics

Consider a random variable X arising from the binomial distribution $\mathrm{Bin}(n, \theta)$, $\theta \in \Theta = [0,1]$. Find the maximum likelihood estimator $\hat{\theta}_{MLE}$ and the Fisher information $I(\theta)$ for $\theta \in \Theta$.

Now consider the following priors on Θ:

(i) a uniform $U([0,1])$ prior on $[0,1]$,

(ii) a prior with density $\pi(\theta)$ proportional to $\sqrt{I(\theta)}$,

(iii) a $\mathrm{Beta}(\sqrt{n}/2, \sqrt{n}/2)$ prior.

Find the means $E[\theta|X]$ and modes $m_\theta|X$ of the posterior distributions corresponding to the prior distributions (i)–(iii). Which of these posterior decision rules coincide with $\hat{\theta}_{MLE}$? Which one is minimax for quadratic risk? Justify your answers.

[You may use the following properties of the $\mathrm{Beta}(a,b)$ $(a>0,\, b>0)$ distribution. Its density $f(x;a,b)$, $x \in [0,1]$, is proportional to $x^{a-1}(1-x)^{b-1}$, its mean is equal to $a/(a+b)$, and its mode is equal to

$$\frac{\max(a-1,0)}{\max(a,1)+\max(b,1)-2}$$

provided either $a>1$ or $b>1$.

You may further use the fact that a unique Bayes rule of constant risk is a unique minimax rule for that risk.]

26K Optimization and Control

As a function of policy π and initial state x, let

$$F(\pi, x) = E_\pi \left[\sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \;\middle|\; x_0 = x \right],$$

where $\beta \geqslant 1$ and $r(x, u) \geqslant 0$ for all x, u. Suppose that for a specific policy π, and all x,

$$F(\pi, x) = \sup_u \left\{ r(x, u) + \beta E[F(\pi, x_1) \mid x_0 = x, u_0 = u] \right\}.$$

Prove that $F(\pi, x) \geqslant F(\pi', x)$ for all π' and x.

A gambler plays games in which he may bet 1 or 2 pounds, but no more than his present wealth. Suppose he has x_t pounds after t games. If he bets i pounds then $x_{t+1} = x_t + i$, or $x_{t+1} = x_t - i$, with probabilities p_i and $1 - p_i$ respectively. Gambling terminates at the first τ such that $x_\tau = 0$ or $x_\tau = 100$. His final reward is $(9/8)^{\tau/2} x_\tau$. Let π be the policy of always betting 1 pound. Given $p_1 = 1/3$, show that $F(\pi, x) \propto x 2^{x/2}$.

Is π optimal when $p_2 = 1/4$?

27K Stochastic Financial Models

(i) What is Brownian motion?

(ii) Suppose that $(B_t)_{t \geqslant 0}$ is Brownian motion, and the price S_t at time t of a risky asset is given by

$$S_t = S_0 \exp\{\, \sigma B_t + (\mu - \tfrac{1}{2}\sigma^2)t \,\}$$

where $\mu > 0$ is the constant growth rate, and $\sigma > 0$ is the constant volatility of the asset. Assuming that the riskless rate of interest is $r > 0$, derive an expression for the price at time 0 of a European call option with strike K and expiry T, explaining briefly the basis for your calculation.

(iii) With the same notation, derive the time-0 price of a European option with expiry T which at expiry pays

$$\{(S_T - K)^+\}^2 / S_T\,.$$

28B Dynamical Systems

(a) An autonomous dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$ has a periodic orbit $\mathbf{x} = \mathbf{X}(t)$ with period T. The linearized evolution of a small perturbation $\mathbf{x} = \mathbf{X}(t) + \boldsymbol{\eta}(t)$ is given by $\eta_i(t) = \Phi_{ij}(t)\eta_j(0)$. Obtain the differential equation and initial condition satisfied by the matrix $\boldsymbol{\Phi}(t)$.

Define the *Floquet multipliers* of the orbit. Explain why one of the multipliers is always unity and give a brief argument to show that the other is given by

$$\exp\left(\int_0^T \nabla \cdot \mathbf{f}(\mathbf{X}(t))\, dt\right).$$

(b) Use the *energy-balance method* for nearly Hamiltonian systems to find leading-order approximations to the two limit cycles of the equation

$$\ddot{x} + \epsilon(2\dot{x}^3 + 2x^3 - 4x^4\dot{x} - \dot{x}) + x = 0,$$

where $0 < \epsilon \ll 1$.

Determine the stability of each limit cycle, giving reasoning where necessary.

[You may assume that $\int_0^{2\pi} \cos^4\theta\, d\theta = 3\pi/4$ and $\int_0^{2\pi} \cos^6\theta\, d\theta = 5\pi/8$.]

29D Integrable Systems

(a) Explain how a vector field

$$V = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$$

generates a 1-parameter group of transformations $g^{\epsilon} : (x,u) \mapsto (\tilde{x},\tilde{u})$ in terms of the solution to an appropriate differential equation. [You may assume the solution to the relevant equation exists and is unique.]

(b) Suppose now that $u = u(x)$. Define what is meant by a *Lie-point symmetry* of the ordinary differential equation

$$\Delta\big[x, u, u^{(1)}, \ldots, u^{(n)}\big] = 0\,, \quad \text{where} \quad u^{(k)} \equiv \frac{d^k u}{dx^k}\,, \quad k = 1, \ldots, n\,.$$

(c) Prove that every homogeneous, linear ordinary differential equation for $u = u(x)$ admits a Lie-point symmetry generated by the vector field

$$V = u\frac{\partial}{\partial u}\,.$$

By introducing new coordinates

$$s = s(x,u), \quad t = t(x,u)$$

which satisfy $V(s) = 1$ and $V(t) = 0$, show that every differential equation of the form

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)u = 0$$

can be reduced to a first-order differential equation for an appropriate function.

30E Partial Differential Equations

Prove that if $\phi \in C(\mathbb{R}^n)$ is absolutely integrable with $\int \phi(x)\, dx = 1$, and $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ for $\epsilon > 0$, then for every Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$ the convolution

$$\phi_\epsilon * f(x) \to f(x)$$

uniformly in x as $\epsilon \downarrow 0$.

Show that the function $N_\epsilon \in C^\infty(\mathbb{R}^3)$ given by

$$N_\epsilon(x) = \frac{1}{4\pi\sqrt{|x|^2 + \epsilon^2}}$$

for $\epsilon > 0$ satisfies

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} -\Delta N_\epsilon(x)\, f(x)\, dx \;=\; f(0)$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Hence prove that the tempered distribution determined by the function $N(x) = (4\pi|x|)^{-1}$ is a fundamental solution of the operator $-\Delta$.

[You may use the fact that $\int_0^\infty r^2/(1+r^2)^{5/2}\, dr = 1/3$.]

31A Principles of Quantum Mechanics

Express the spin operator $\mathbf{S}$ for a particle of spin $\frac{1}{2}$ in terms of the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Show that $(\mathbf{n} \cdot \boldsymbol{\sigma})^2 = \mathbb{I}$ for any unit vector $\mathbf{n}$ and deduce that

$$e^{-i\theta\, \mathbf{n}\cdot\mathbf{S}/\hbar} = \mathbb{I} \cos(\theta/2) - i(\mathbf{n} \cdot \boldsymbol{\sigma}) \sin(\theta/2) .$$

The space of states V for a particle of spin $\frac{1}{2}$ has basis states $|\uparrow\rangle$, $|\downarrow\rangle$ which are eigenstates of S_3 with eigenvalues $\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$ respectively. If the Hamiltonian for the particle is $H = \frac{1}{2}\alpha\hbar\sigma_1$, find

$$e^{-itH/\hbar}|\uparrow\rangle \qquad \text{and} \qquad e^{-itH/\hbar}|\downarrow\rangle$$

as linear combinations of the basis states.

The space of states for a system of two spin $\frac{1}{2}$ particles is $V \otimes V$. Write down explicit expressions for the joint eigenstates of $\mathbf{J}^2$ and J_3, where $\mathbf{J}$ is the sum of the spin operators for the particles.

Suppose that the two-particle system has Hamiltonian $H = \frac{1}{2}\lambda\hbar(\sigma_1 \otimes \mathbb{I} - \mathbb{I} \otimes \sigma_1)$ and that at time $t = 0$ the system is in the state with J_3 eigenvalue $\hbar$. Calculate the probability that at time $t > 0$ the system will be measured to be in the state with $\mathbf{J}^2$ eigenvalue zero.

32A Applications of Quantum Mechanics

A beam of particles of mass m and energy $\hbar^2 k^2/2m$ is incident on a target at the origin described by a spherically symmetric potential $V(r)$. Assuming the potential decays rapidly as $r \to \infty$, write down the asymptotic form of the wavefunction, defining the scattering amplitude $f(\theta)$.

Consider a free particle with energy $\hbar^2 k^2/2m$. State, without proof, the general axisymmetric solution of the Schrödinger equation for $r > 0$ in terms of spherical Bessel and Neumann functions j_ℓ and n_ℓ, and Legendre polynomials P_ℓ ($\ell = 0, 1, 2, \ldots$). Hence define the partial wave phase shifts δ_ℓ for scattering from a potential $V(r)$ and derive the partial wave expansion for $f(\theta)$ in terms of phase shifts.

Now suppose

$$V(r) = \begin{cases} \hbar^2\gamma^2/2m & r < a \\ 0 & r > a \end{cases}$$

with $\gamma > k$. Show that the S-wave phase shift δ_0 obeys

$$\frac{\tanh(\kappa a)}{\kappa a} = \frac{\tan(ka + \delta_0)}{ka}$$

where $\kappa^2 = \gamma^2 - k^2$. Deduce that for an S-wave solution

$$f \to \frac{\tanh \gamma a - \gamma a}{\gamma} \qquad \text{as} \qquad k \to 0 \, .$$

[You may assume : $\quad \exp(ikr\cos\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, i^\ell\, j_\ell(kr)\, P_\ell(\cos\theta)$

and $\quad j_\ell(\rho) \sim \frac{1}{\rho}\sin(\rho - \ell\pi/2)\ , \quad n_\ell(\rho) \sim -\frac{1}{\rho}\cos(\rho - \ell\pi/2) \quad \text{as} \quad \rho \to \infty \, .$]

33C Statistical Physics

(a) State the Bose–Einstein distribution formula for the mean occupation numbers n_i of discrete single-particle states i with energies E_i in a gas of bosons. Write down expressions for the total particle number N and the total energy U when the single-particle states can be treated as continuous, with energies $E \geqslant 0$ and density of states $g(E)$.

(b) Blackbody radiation at temperature T is equivalent to a gas of photons with

$$g(E) = AVE^2$$

where V is the volume and A is a constant. What value of the chemical potential is required when applying the Bose–Einstein distribution to photons? Show that the heat capacity at constant volume satisfies $C_V \propto T^\alpha$ for some constant α, to be determined.

(c) Consider a system of bosonic particles of fixed total number $N \gg 1$. The particles are trapped in a potential which has ground state energy zero and which gives rise to a density of states $g(E) = BE^2$, where B is a constant. Explain, for this system, what is meant by Bose–Einstein condensation and show that the critical temperature satisfies $T_c \propto N^{1/3}$. If N_0 is the number of particles in the ground state, show that for T just below T_c

$$N_0/N \approx 1 - (T/T_c)^\gamma$$

for some constant γ, to be determined.

(d) Would you expect photons to exhibit Bose–Einstein condensation? Explain your answer very briefly.

34D General Relativity

(a) The Schwarzschild metric is

$$ds^2 = -(1 - r_s/r)dt^2 + (1 - r_s/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(in units for which the speed of light $c = 1$). Show that a timelike geodesic in the equatorial plane obeys

$$\tfrac{1}{2}\dot{r}^2 + V(r) = \tfrac{1}{2}E^2\,,$$

where

$$2V(r) = \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{h^2}{r^2}\right)$$

and E and h are constants.

(b) For a circular orbit of radius r, show that

$$h^2 = \frac{r^2 r_s}{2r - 3r_s}\,.$$

Given that the orbit is stable, show that $r > 3r_s$.

(c) Alice lives on a small planet that is in a stable circular orbit of radius r around a (non-rotating) black hole of radius r_s. Bob lives on a spacecraft in deep space far from the black hole and at rest relative to it. Bob is ageing k times faster than Alice. Find an expression for k^2 in terms of r and r_s and show that $k < \sqrt{2}$.

35E Fluid Dynamics II

Consider an infinite rigid cylinder of radius a parallel to a horizontal rigid stationary surface. Let $\mathbf{e}_x$ be the direction along the surface perpendicular to the cylinder axis, $\mathbf{e}_y$ the direction normal to the surface (the surface is at $y = 0$) and $\mathbf{e}_z$ the direction along the axis of the cylinder. The cylinder moves with constant velocity $U\mathbf{e}_x$. The minimum separation between the cylinder and the surface is denoted by $h_0 \ll a$.

(i) What are the conditions for the flow in the thin gap between the cylinder and the surface to be described by the lubrication equations? State carefully the relevant length scale in the $\mathbf{e}_x$ direction.

(ii) Without doing any calculation, explain carefully why, in the lubrication limit, the net fluid force $\mathbf{F}$ acting on the stationary surface at $y = 0$ has no component in the $\mathbf{e}_y$ direction.

(iii) Using the lubrication approximation, calculate the $\mathbf{e}_x$ component of the velocity field in the gap between the cylinder and the surface, and determine the pressure gradient as a function of the gap thickness $h(x)$.

(iv) Compute the tangential component of the force, $\mathbf{e}_x \cdot \mathbf{F}$, acting on the bottom surface per unit length in the $\mathbf{e}_z$ direction.

[You may quote the following integrals:

$$\int_{-\infty}^{\infty} \frac{du}{(1+u^2)} = \pi, \quad \int_{-\infty}^{\infty} \frac{du}{(1+u^2)^2} = \frac{\pi}{2}, \quad \int_{-\infty}^{\infty} \frac{du}{(1+u^2)^3} = \frac{3\pi}{8}. \quad]$$

36B Waves

A uniform elastic solid with density ρ and Lamé moduli λ and μ occupies the region between rigid plane boundaries $z = 0$ and $z = h$. Starting with the linear elastic wave equation, show that SH waves can propagate in the x-direction within this waveguide, and find the dispersion relation $\omega(k)$ for the various modes.

State the cut-off frequency for each mode. Find the corresponding phase velocity $c(k)$ and group velocity $c_g(k)$, and sketch these functions for $k, \omega > 0$.

Define the time and cross-sectional average appropriate for a mode with frequency ω. Show that for each mode the average kinetic energy is equal to the average elastic energy. [You may assume that the elastic energy per unit volume is $\frac{1}{2}(\lambda e_{kk}^2 + 2\mu e_{ij}e_{ij})$.]

An elastic displacement of the form $\mathbf{u} = (0, f(x, z), 0)$ is created in a region near $x = 0$, and then released at $t = 0$. Explain briefly how the amplitude of the resulting disturbance varies with time as $t \to \infty$ at the moving position $x = Vt$ for each of the cases $0 < V^2 < \mu/\rho$ and $V^2 > \mu/\rho$. [You may quote without proof any generic results from the method of stationary phase.]

37E Numerical Analysis

(a) The boundary value problem $-\Delta u + cu = f$ on the unit square $[0,1]^2$ with zero boundary conditions and scalar constant $c > 0$ is discretised using finite differences as

$$-u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} + 4u_{i,j} + ch^2 u_{i,j} = h^2 f(ih, jh), \qquad i,j = 1,\ldots,m,$$

with $h = 1/(m+1)$. Show that for the resulting system $Au = b$, for a suitable matrix A and vectors u and b, both the Jacobi and Gauss–Seidel methods converge. [You may cite and use known results on the discretised Laplace operator and on the convergence of iterative methods.]

Define the Jacobi method with relaxation parameter ω. Find the eigenvalues $\lambda_{k,l}$ of the iteration matrix H_ω for the above problem and show that, in order to ensure convergence for all h, the condition $0 < \omega \leqslant 1$ is necessary.

[*Hint: The eigenvalues of the discretised Laplace operator in two dimensions are* $4\left(\sin^2 \frac{\pi kh}{2} + \sin^2 \frac{\pi lh}{2}\right)$ *for integers* k, l.]

(b) Explain the components and steps in a multigrid method for solving the Poisson equation, discretised as $A_h u_h = b_h$. If we use the relaxed Jacobi method within the multigrid method, is it necessary to choose $\omega \neq 1$ to get fast convergence? Explain why or why not.

END OF PAPER