UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 5 June, 2015 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Define the *dual space* V^* of a vector space V. Given a basis $\{x_1, \ldots, x_n\}$ of V define its *dual* and show it is a basis of V^*.

Let V be a 3-dimensional vector space over $\mathbb{R}$ and let $\{\zeta_1, \zeta_2, \zeta_3\}$ be the basis of V^* dual to the basis $\{x_1, x_2, x_3\}$ for V. Determine, in terms of the ζ_i, the bases dual to each of the following:

(a) $\{x_1 + x_2, x_2 + x_3, x_3\}$,

(b) $\{x_1 + x_2, x_2 + x_3, x_3 + x_1\}$.

2F Groups, Rings and Modules

Let R be a commutative ring. Define what it means for an ideal $I \subseteq R$ to be *prime*. Show that $I \subseteq R$ is prime if and only if R/I is an integral domain.

Give an example of an integral domain R and an ideal $I \subset R$, $I \neq R$, such that R/I is not an integral domain.

3G Analysis II

Define what is meant for two norms on a vector space to be *Lipschitz equivalent*.

Let $C_c^1([-1,1])$ denote the vector space of continuous functions $f : [-1,1] \to \mathbb{R}$ with continuous first derivatives and such that $f(x) = 0$ for x in some neighbourhood of the end-points -1 and 1. Which of the following four functions $C_c^1([-1,1]) \to \mathbb{R}$ define norms on $C_c^1([-1,1])$ (give a brief explanation)?

$$p(f) = \sup |f|, \qquad q(f) = \sup(|f| + |f'|),$$

$$r(f) = \sup |f'|, \qquad s(f) = \left| \int_{-1}^{1} f(x)dx \right|.$$

Among those that define norms, which pairs are Lipschitz equivalent? Justify your answer.

4G Complex Analysis

Let f be a continuous function defined on a connected open set $D \subset \mathbb{C}$. Prove carefully that the following statements are equivalent.

(i) There exists a holomorphic function F on D such that $F'(z) = f(z)$.

(ii) $\int_\gamma f(z)dz = 0$ holds for every closed curve γ in D.

5C Methods

(a) The convolution $f * g$ of two functions $f,\ g : \mathbb{R} \to \mathbb{C}$ is related to their Fourier transforms $\tilde{f},\ \tilde{g}$ by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{g}(k)e^{ikx}\,dk = \int_{-\infty}^{\infty} f(u)g(x-u)\,du\,.$$

Derive Parseval's theorem for Fourier transforms from this relation.

(b) Let $a > 0$ and

$$f(x) = \begin{cases} \cos x & \text{for } \ x \in [-a, a] \\ 0 & \text{elsewhere.} \end{cases}$$

(i) Calculate the Fourier transform $\tilde{f}(k)$ of $f(x)$.

(ii) Determine how the behaviour of $\tilde{f}(k)$ in the limit $|k| \to \infty$ depends on the value of a. Briefly interpret the result.

6D Quantum Mechanics

The radial wavefunction $R(r)$ for an electron in a hydrogen atom satisfies the equation

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R(r)\right) + \frac{\hbar^2}{2mr^2}\ell(\ell+1)R(r) - \frac{e^2}{4\pi\epsilon_0 r}R(r) = E\,R(r) \qquad (*)$$

Briefly explain the origin of each term in this equation.

The wavefunctions for the ground state and the first radially excited state, both with $\ell = 0$, can be written as

$$\begin{aligned} R_1(r) &= N_1 e^{-\alpha r} \\ R_2(r) &= N_2\left(1 - \tfrac{1}{2}r\alpha\right)e^{-\frac{1}{2}\alpha r} \end{aligned}$$

where N_1 and N_2 are normalisation constants. Verify that $R_1(r)$ is a solution of $(*)$, determining α and finding the corresponding energy eigenvalue E_1. Assuming that $R_2(r)$ is a solution of $(*)$, compare coefficients of the dominant terms when r is large to determine the corresponding energy eigenvalue E_2. [You do *not* need to find N_1 or N_2, nor show that R_2 is a solution of $(*)$.]

A hydrogen atom makes a transition from the first radially excited state to the ground state, emitting a photon. What is the angular frequency of the emitted photon?

7A Electromagnetism

From Maxwell's equations, derive the Biot–Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, d^3\mathbf{r}',$$

giving the magnetic field $\mathbf{B}(\mathbf{r})$ produced by a steady current density $\mathbf{J}(\mathbf{r})$ that vanishes outside a bounded region V.

[You may assume that you can choose a gauge such that the divergence of the magnetic vector potential is zero.]

8D Numerical Analysis

Given $n+1$ distinct points $\{x_0, x_1, \ldots, x_n\}$, let $p_n \in \mathbb{P}_n$ be the real polynomial of degree n that interpolates a continuous function f at these points. State the *Lagrange interpolation formula.*

Prove that p_n can be written in the *Newton form*

$$p_n(x) = f(x_0) + \sum_{k=1}^{n} f\,[x_0, \ldots, x_k] \prod_{i=0}^{k-1} (x - x_i)\,,$$

where $f[x_0, \ldots, x_k]$ is the *divided difference*, which you should define. [An explicit expression for the divided difference is *not* required.]

Explain why it can be more efficient to use the Newton form rather than the Lagrange formula.

9H Markov Chains

Let $X_0, X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p$, $0 < p < 1$. Let $Z_n = X_{n-1} + cX_n$, $n = 1, 2, \ldots$, where c is a constant. For each of the following cases, determine whether or not $(Z_n : n \geqslant 1)$ is a Markov chain:

(a) $c = 0$;

(b) $c = 1$;

(c) $c = 2$.

In each case, if $(Z_n : n \geqslant 1)$ is a Markov chain, explain why, and give its state space and transition matrix; if it is not a Markov chain, give an example to demonstrate that it is not.

SECTION II

10E Linear Algebra

Suppose U and W are subspaces of a vector space V. Explain what is meant by $U \cap W$ and $U + W$ and show that both of these are subspaces of V.

Show that if U and W are subspaces of a finite dimensional space V then

$$\dim U + \dim W = \dim(U \cap W) + \dim(U + W).$$

Determine the dimension of the subspace W of $\mathbb{R}^5$ spanned by the vectors

$$\begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 5 \\ -1 \\ -1 \end{pmatrix}.$$

Write down a 5×5 matrix which defines a linear map $\mathbb{R}^5 \to \mathbb{R}^5$ with $(1,1,1,1,1)^T$ in the kernel and with image W.

What is the dimension of the space spanned by all linear maps $\mathbb{R}^5 \to \mathbb{R}^5$
(i) with $(1,1,1,1,1)^T$ in the kernel and with image contained in W,
(ii) with $(1,1,1,1,1)^T$ in the kernel or with image contained in W?

11F Groups, Rings and Modules

Find $a \in \mathbb{Z}_7$ such that $\mathbb{Z}_7[x]/(x^3 + a)$ is a field F. Show that for your choice of a, every element of $\mathbb{Z}_7$ has a cube root in the field F.

Show that if F is a finite field, then the multiplicative group $F^\times = F \setminus \{0\}$ is cyclic.

Show that $F = \mathbb{Z}_2[x]/(x^3 + x + 1)$ is a field. How many elements does F have? Find a generator for $F^\times$.

12G Analysis II

Consider the space ℓ^∞ of bounded real sequences $x = (x_i)_{i=1}^\infty$ with the norm $\|x\|_\infty = \sup_i |x_i|$. Show that for every bounded sequence $x^{(n)}$ in ℓ^∞ there is a subsequence $x^{(n_j)}$ which converges in every coordinate, i.e. the sequence $(x_i^{(n_j)})_{j=1}^\infty$ of real numbers converges for each i. Does every bounded sequence in ℓ^∞ have a convergent subsequence? Justify your answer.

Let $\ell^1 \subset \ell^\infty$ be the subspace of real sequences $x = (x_i)_{i=1}^\infty$ such that $\sum_{i=1}^\infty |x_i|$ converges. Is ℓ^1 complete in the norm $\|\cdot\|_\infty$ (restricted from ℓ^∞ to ℓ^1)? Justify your answer.

Suppose that (x_i) is a real sequence such that, for every $(y_i) \in \ell^\infty$, the series $\sum_{i=1}^\infty x_i y_i$ converges. Show that $(x_i) \in \ell^1$.

Suppose now that (x_i) is a real sequence such that, for every $(y_i) \in \ell^1$, the series $\sum_{i=1}^\infty x_i y_i$ converges. Show that $(x_i) \in \ell^\infty$.

13E Metric and Topological Spaces

Explain what it means for a metric space (M, d) to be (i) *compact*, (ii) *sequentially compact*. Prove that a compact metric space is sequentially compact, stating clearly any results that you use.

Let (M, d) be a compact metric space and suppose $f\colon M \to M$ satisfies $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. Prove that f is surjective, stating clearly any results that you use. [*Hint: Consider the sequence* $(f^n(x))$ *for* $x \in M$.]

Give an example to show that the result does not hold if M is not compact.

14B Complex Methods

(i) State and prove the convolution theorem for Laplace transforms of two real-valued functions.

(ii) Let the function $f(t)$, $t \geqslant 0$, be equal to 1 for $0 \leqslant t \leqslant a$ and zero otherwise, where a is a positive parameter. Calculate the Laplace transform of f. Hence deduce the Laplace transform of the convolution $g = f * f$. Invert this Laplace transform to obtain an explicit expression for $g(t)$.

[*Hint: You may use the notation* $(t-a)_+ = H(t-a) \cdot (t-a)$.]

15F Geometry

Let $\alpha(s) = \big(f(s), g(s)\big)$ be a curve in $\mathbb{R}^2$ parameterized by arc length, and consider the surface of revolution S in $\mathbb{R}^3$ defined by the parameterization

$$\sigma(u,v) = \big(f(u)\cos v, f(u)\sin v, g(u)\big).$$

In what follows, you may use that a curve $\sigma \circ \gamma$ in S, with $\gamma(t) = \big(u(t), v(t)\big)$, is a geodesic if and only if

$$\ddot{u} = f(u)\frac{df}{du}\dot{v}^2, \quad \frac{d}{dt}\big(f(u)^2\dot{v}\big) = 0.$$

(i) Write down the first fundamental form for S, and use this to write down a formula which is equivalent to $\sigma \circ \gamma$ being a unit speed curve.

(ii) Show that for a given u_0, the circle on S determined by $u = u_0$ is a geodesic if and only if $\frac{df}{du}(u_0) = 0$.

(iii) Let $\gamma(t) = \big(u(t), v(t)\big)$ be a curve in $\mathbb{R}^2$ such that $\sigma \circ \gamma$ parameterizes a unit speed curve that is a geodesic in S. For a given time t_0, let $\theta(t_0)$ denote the angle between the curve $\sigma \circ \gamma$ and the circle on S determined by $u = u(t_0)$. Derive *Clairault's relation* that

$$f\big(u(t)\big)\cos\big(\theta(t)\big)$$

is independent of t.

16A Variational Principles

Derive the Euler–Lagrange equation for the integral

$$\int_{x_0}^{x_1} f(x,u,u')\; dx$$

where $u(x_0)$ is allowed to float, $\partial f/\partial u'|_{x_0} = 0$ and $u(x_1)$ takes a given value.

Given that $y(0)$ is finite, $y(1) = 1$ and $y'(1) = 1$, find the stationary value of

$$J = \int_0^1 \Big(x^4(y'')^2 + 4x^2(y')^2\Big)\; dx.$$

17C Methods

Describe the method of characteristics to construct solutions for 1st-order, homogeneous, linear partial differential equations

$$\alpha(x,y)\frac{\partial u}{\partial x} + \beta(x,y)\frac{\partial u}{\partial y} = 0\,,$$

with initial data prescribed on a curve $x_0(\sigma)$, $y_0(\sigma)$: $u(x_0(\sigma), y_0(\sigma)) = h(\sigma)$.

Consider the partial differential equation (here the two independent variables are time t and spatial direction x)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0\,,$$

with initial data $u(t=0,x) = e^{-x^2}$.

(i) Calculate the characteristic curves of this equation and show that u remains constant along these curves. Qualitatively sketch the characteristics in the (x,t) diagram, i.e. the x axis is the horizontal and the t axis is the vertical axis.

(ii) Let $\tilde{x}_0$ denote the x value of a characteristic at time $t = 0$ and thus label the characteristic curves. Let $\tilde{x}$ denote the x value at time t of a characteristic with given $\tilde{x}_0$. Show that $\partial\tilde{x}/\partial\tilde{x}_0$ becomes a non-monotonic function of $\tilde{x}_0$ (at fixed t) at times $t > \sqrt{e/2}$, i.e. $\tilde{x}(\tilde{x}_0)$ has a local minimum or maximum. Qualitatively sketch snapshots of the solution $u(t,x)$ for a few fixed values of $t \in [0, \sqrt{e/2}]$ and briefly interpret the onset of the non-monotonic behaviour of $\tilde{x}(\tilde{x}_0)$ at $t = \sqrt{e/2}$.

18B Fluid Dynamics

Consider a steady inviscid, incompressible fluid of constant density ρ in the absence of external body forces. A cylindrical jet of area A and speed U impinges fully on a stationary sphere of radius R with $A < \pi R^2$. The flow is assumed to remain axisymmetric and be deflected into a conical sheet of vertex angle $\alpha > 0$.

(i) Show that the speed of the fluid in the conical sheet is constant.

(ii) Use conservation of mass to show that the width $d(r)$ of the fluid sheet at a distance $r \gg R$ from point of impact is given by

$$d(r) = \frac{A}{2\pi r \sin\alpha}.$$

(iii) Use Euler's equation to derive the momentum integral equation

$$\iint_S (p n_i + \rho n_j u_j u_i)\, dS = 0,$$

for a closed surface S with normal $\mathbf{n}$ where u_m is the mth component of the velocity field in cartesian coordinates and p is the pressure.

(iv) Use the result from (iii) to calculate the net force on the sphere.

19H Statistics

Consider a linear model $\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\mathbf{Y}$ is an $n \times 1$ vector of observations, X is a known $n \times p$ matrix, $\boldsymbol{\beta}$ is a $p \times 1$ $(p < n)$ vector of unknown parameters and $\boldsymbol{\varepsilon}$ is an $n \times 1$ vector of independent normally distributed random variables each with mean zero and unknown variance σ^2. Write down the log-likelihood and show that the maximum likelihood estimators $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$ of $\boldsymbol{\beta}$ and σ^2 respectively satisfy

$$X^T X \hat{\boldsymbol{\beta}} = X^T \mathbf{Y}, \qquad \frac{1}{\hat{\sigma}^4}(\mathbf{Y} - X\hat{\boldsymbol{\beta}})^T(\mathbf{Y} - X\hat{\boldsymbol{\beta}}) = \frac{n}{\hat{\sigma}^2}$$

(^{T} denotes the transpose). Assuming that $X^T X$ is invertible, find the solutions $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$ of these equations and write down their distributions.

Prove that $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$ are independent.

Consider the model $Y_{ij} = \mu_i + \gamma x_{ij} + \varepsilon_{ij}$, $i = 1, 2, 3$ and $j = 1, 2, 3$. Suppose that, for all i, $x_{i1} = -1$, $x_{i2} = 0$ and $x_{i3} = 1$, and that ε_{ij}, $i, j = 1, 2, 3$, are independent $N(0, \sigma^2)$ random variables where σ^2 is unknown. Show how this model may be written as a linear model and write down $\mathbf{Y}$, X, $\boldsymbol{\beta}$ and $\boldsymbol{\varepsilon}$. Find the maximum likelihood estimators of μ_i $(i = 1, 2, 3)$, γ and σ^2 in terms of the Y_{ij}. Derive a $100(1-\alpha)\%$ confidence interval for σ^2 and for $\mu_2 - \mu_1$.

[You may assume that, if $\mathbf{W} = (\mathbf{W_1}^T, \mathbf{W_2}^T)^T$ is multivariate normal with $\mathrm{cov}(\mathbf{W_1}, \mathbf{W_2}) = 0$, then $\mathbf{W_1}$ and $\mathbf{W_2}$ are independent.]

20H Optimization

Suppose the recycling manager in a particular region is responsible for allocating all the recyclable waste that is collected in n towns in the region to the m recycling centres in the region. Town i produces s_i lorry loads of recyclable waste each day, and recycling centre j needs to handle d_j lorry loads of waste a day in order to be viable. Suppose that $\sum_i s_i = \sum_j d_j$. Suppose further that c_{ij} is the cost of transporting a lorry load of waste from town i to recycling centre j. The manager wishes to decide the number x_{ij} of lorry loads of recyclable waste that should go from town i to recycling centre j, $i = 1, \ldots, n$, $j = 1, \ldots, m$, in such a way that all the recyclable waste produced by each town is transported to recycling centres each day, and each recycling centre works exactly at the viable level each day. Use the Lagrangian sufficiency theorem, which you should quote carefully, to derive necessary and sufficient conditions for (x_{ij}) to minimise the total cost under the above constraints.

Suppose that there are three recycling centres A, B and C, needing 5, 20 and 20 lorry loads of waste each day, respectively, and suppose there are three towns a, b and c producing 20, 15 and 10 lorry loads of waste each day, respectively. The costs of transporting a lorry load of waste from town a to recycling centres A, B and C are £90, £100 and £100, respectively. The corresponding costs for town b are £130, £140 and £100, while for town c they are £110, £80 and £80. Recycling centre A has reported that it currently receives 5 lorry loads of waste per day from town a, and recycling centre C has reported that it currently receives 10 lorry loads of waste per day from each of towns b and c. Recycling centre B has failed to report. What is the cost of the current arrangement for transporting waste from the towns to the recycling centres? Starting with the current arrangement as an initial solution, use the transportation algorithm (explaining each step carefully) in order to advise the recycling manager how many lorry loads of waste should go from each town to each of the recycling centres in order to minimise the cost. What is the minimum cost?

END OF PAPER